Mathematics - Course 321

LOGARITHMS AND EXPONENTIALS

I INTRODUCTION

(a) Exponential Functions

DEFINITION:

An *exponential function* is a function of the form $f(x) = a^x$, where "a" is a real positive constant.

The distinction between the exponential function, a^x, and the more familiar power function, x^a, should be clear from the following example in which a = 2:

Example 1:

Plot on the same graph the functions $y = 2^x$ and $y = x^2$ over the domain $-4 \leqslant x \leqslant 4$.

x	-4	-3	-2	-1	0	1	2	3	4
2^x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
x^2	16	9	4	1	0	1	4	9	16

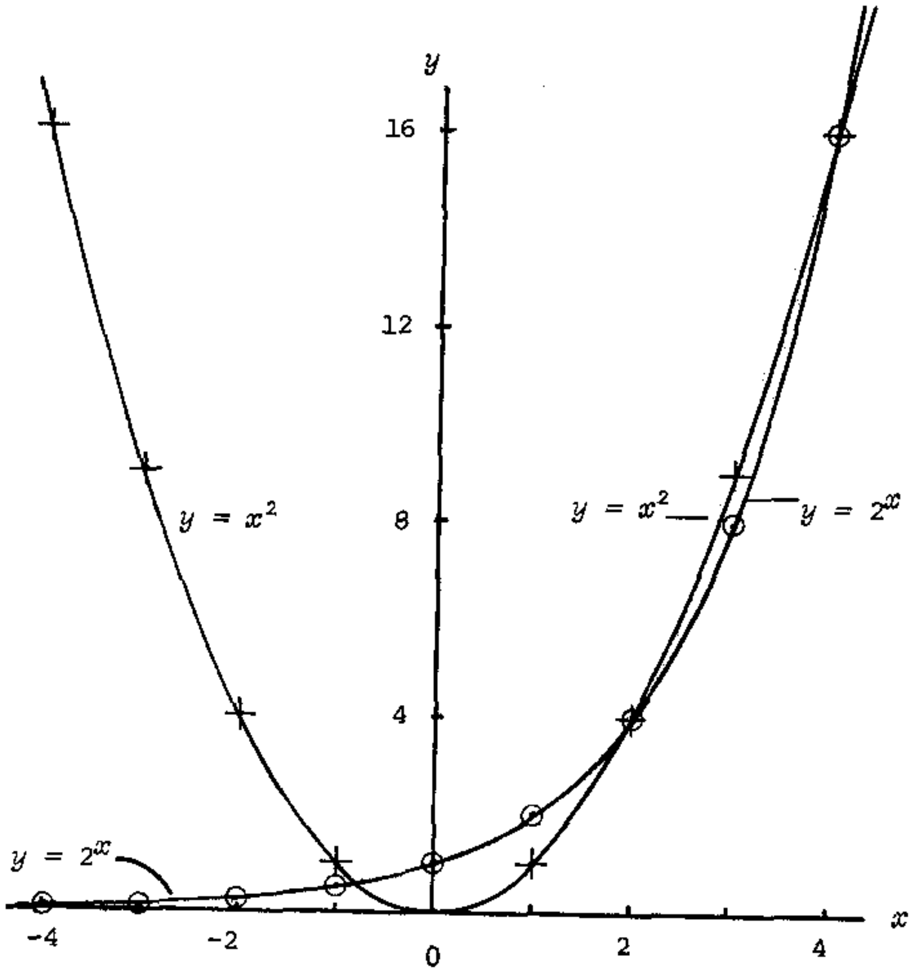


Figure 1

Note that the curve $y = 2^x$ is *asymptotic* to the negative x-axis, ie, the curve approaches ever more closely to the negative x-axis as the magnitude of x grows, but never actually reaches the axis for any finite x-value. The curve $y = 2^{-x}$ is the mirror image in the y-axis of $y = 2^x$, and is asymptotic to the positive x-axis. (Check this.)

Thus, in general, if $a > 1$, the exponential functions a^x and a^{-x} have the characteristic shapes illustrated in Figure 2.

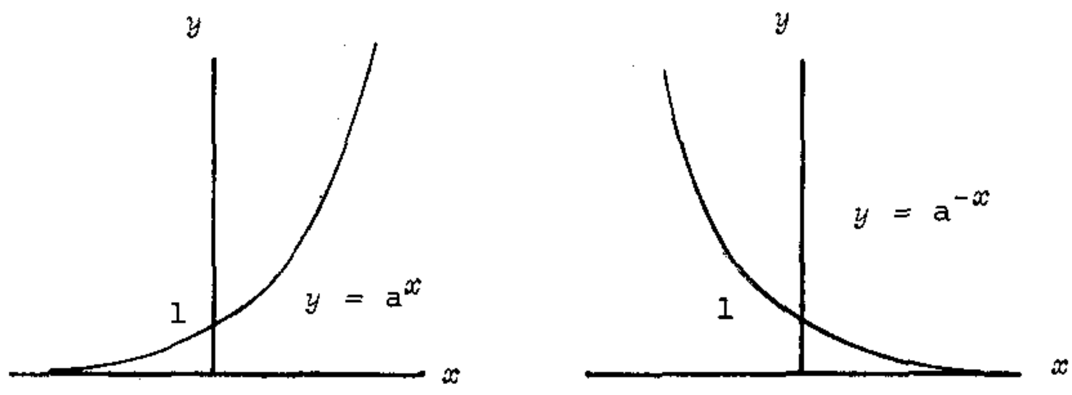


Figure 2

(b) Logarithmic Functions

DEFINITION:

The *logarithm of x to the base "a"* designated "$\log_a x$", is the exponent to which "a" must be raised to produce x.

That is: $\log_a x = y \Longleftrightarrow a^y = x$

eg, $\log_3 9 = 2$, since $3^2 = 9$

eg, $\log_2 64 = 6$, since $2^6 = 64$

In general, the curve $y = \log_a x$ has the characteristic shape shown in Figure 3.

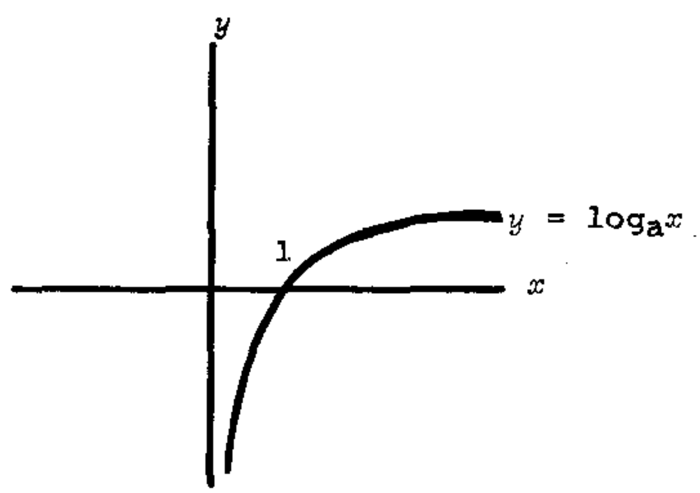


Figure 3

The bases 10 and e are so commonly used as to justify log x and $\ln x$ functions on scientific calculators. These are the so-called:

(1) *Common Logarithms*, to base 10, and

(2) *Natural Logarithms*, to base "e".

(e = 2.718281828...)

The rationale behind the special provisions for common logarithms is our use of the decimal system (base 10), while the rationale for natural logarithms is the fact that exponential functions (base "e") are abundant in technical applications.

eg: $V(t) = V_0 e^{-t/RC}$, where

Vo	=	initial voltage at t=o
V(t)	=	voltage at time t
t	=	time in seconds
R	=	resistance
C	=	capacitance

The above definition for $\log_a x$ is restated here specifically for common and natural logarithms:

DEFINITION:

The *common logarithm of x*, designated "$\log_{10} x$" (or simply "log x"), is the exponent to which 10 must be raised to produce x.

eg, $\log 1000 = 3$, since $10^3 = 1000$

eg, $\log \sqrt{10} = \frac{1}{2}$, since $10^{\frac{1}{2}} = \sqrt{10}$

DEFINITION:

The *natural logarithm of x*, designated "$\log_e x$" (or simply "$\ln x$"), is the exponent to which "e" must be raised to produce x.

eg, $\ln e^5 = 5$, since $e^5 = e^5$

eg, $\ln \sqrt[3]{e} = \frac{1}{3}$, since $e^{1/3} = \sqrt[3]{e}$

II USE OF LOGARITHMS IN COMPUTATION OF COMPLEX ARITHMETIC EXPRESSIONS

Logarithms are used to reduce the operations of multiplication, division, and exponentiation to addition, subtraction, and multiplication, respectively, according to the following laws:

LAW 1: $\log XY = \log X + \log Y$

LAW 2: $\log \frac{X}{Y} = \log X - \log Y$

LAW 3: $\log X^n = n \log X$

With the introduction of the scientific calculator, the computation of complex arithmetic expressions has been greatly simplified. In fact, the use of logarithms in their evaluation has been rendered virtually obsolete. However, the trainee should become fully familiar with the laws governing the use of logarithms as an aid in solving some types of problems which will be introduced later in this lesson. To this end, several examples are now presented which illustrate the use of logarithms in the evaluation of complex arithmetic and algebraic expressions.

Example 1:

Evaluate $\sqrt[5]{(0.007294)^3}$

Solution:

(a) Modern calculator technique (use of Y^x)

$$\begin{aligned}\sqrt[5]{(0.007294)^3} &= (0.007294)^{3/5}\\ &= (0.007294)^{0.6}\\ &= 0.05221\end{aligned}$$

(b) Use of logarithms (obsolete method)

$$\begin{aligned}\text{Let } x &= \sqrt[5]{(0.007294)^3}\\ \text{then } \log x &= \log (0.007294)^{3/5}\\ &= \frac{3}{5} \log (0.007294)\\ &= \frac{3}{5} (-2.1370)\\ &= -1.2822\\ \therefore \log x &= -1.2822\end{aligned}$$

How does one now find the value of x?

Recall that log x, by definition, is the exponent to which 10 is raised to produce x. Thus,

$$x = 10^{-1.2822}$$

(This process of exponentiating to find x is also called taking the antilogarithm, to base 10, of -1.2822.)

Example 2

Evaluate $$\frac{(7.236)^{1/3} \times (4.36)^2}{(0.00287)^4}$$

Solution:

$$\text{Let} \quad x = \frac{(7.236)^{1/3} \times (4.36)^2}{(0.00287)^4}$$

$$\begin{aligned}
\text{Then log} \quad x &= \log \left[\frac{(7.236)^{1/3} \times (4.36)^2}{(0.00287)^4}\right] \\
&= \log \left[(7.236)^{1/3} \times (4.36)^2\right] - \log(0.00287)^4 \\
&= \log \quad (7.236)^{1/3} + \log(4.36)^2 - \log(0.00287)^4 \\
&= \frac{1}{3} \log 7.236 + 2 \log 4.36 - 4 \log(0.00287) \\
&= \frac{1}{3} (0.8595) + 2(0.6395) - 4(-2.5421) \\
&= 11.7339
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \log x &= 11.7339 \\
x &= 10^{11.7339} \\
&= 5.42 \times 10^{11}
\end{aligned}$$

Example 3:

Express $\sqrt[3]{\dfrac{X^2\ \sqrt{Y}}{Z^5}}$ in terms of log X, log Y and log Z.

Solution:

$$\log \sqrt[3]{\frac{X^2\ \sqrt{Y}}{Z^5}} = \log\left(\frac{X^2\ \sqrt{Y}}{Z^5}\right)^{1/3} \qquad \text{(power law: } x^{1/n} = \sqrt[n]{x}\text{)}$$

$$= \frac{1}{3}\log\left(\frac{X^2\ \sqrt{Y}}{Z^5}\right) \qquad \text{(Law 3)}$$

$$= \frac{1}{3}\left[\log\ (X^2\ \sqrt{Y}) - \log Z^5\right] \qquad \text{(Law 2)}$$

$$= \frac{1}{3}\left[\log X^2 + \log \sqrt{Y} - \log Z^5\right] \qquad \text{(Law 1)}$$

$$= \frac{1}{3}\left[2 \log X + \frac{1}{2}\log Y - 5 \log Z\right] \qquad \text{(Law 3)}$$

III CONNECTION BETWEEN EXPONENTIALS AND LOGARITHMS

Taking the logarithm of x to base "a" and raising "a" to the exponent x are opposite operations in the same sense that multiplication and division are opposite operations, ie, the one operation 'undoes' the other.

For example, any one of the following sequences of operations on x will give x back again as the final result:

(1) first multiply by 2, then divide result by 2, ie, $(2x) \div 2 = x$.

(2) first divide by 2, then multiply result by 2, ie, $(x\div 2)\ (2) = x$.

(3) first take logarithm to base 2, then raise 2 to the result, ie $2^{\log_2 x} = x$.

(4) first raise 2 to exponent x, then take logarithm of result to base 2; ie, $\log_2 2^x = x$.

The above explanation can also be presented in tabular form.

Table Illustrating the Effect of Applying Opposite Operations Consecutively

Start With	x	x	x	x
First Operation	add k	multiply by k	exponentiate with base a	take log to base a
Interim Result	$x + k$	kx	a^x	$\log_a x$
Second (Opposite) Operation	subtract k	divide by k	take log to base a	exponentiate with base a (ANTILOG)
Final Result	$(x+k)-k=x$	$(kx)\div k=x$	$\log_a a^x=x$	$a^{\log_a x}=x$
Example	$(x+2)-2=x$	$(2x)\div 2=x$	$\log_2 2^x=x$	$2^{\log_2 x}=x$

The connection between logarithms and exponentials can be further summarized as follows:

$$\boxed{\log_a a^x = x = a^{\log_a x}}$$

The corresponding statements for common and natural logarithms are:

$\log 10^x = x = 10^{\log x}$ (common logs)

$\ln e^x = x = e^{\ln x}$ (natural logs)

At this point the trainee should be able to evaluate simple expressions involving logarithms, without recourse to aids, by applying the foregoing definitions.

Example 4:

Evaluate without recourse to aids: $5^{\log_5 x}$

Solution:

By definition, $\log_5 x$ represents the number to which 5 must be raised to produce x. Therefore, in the above expression, 5 is being raised to that number which will produce x.

ie: $5^{\log_5 x} = x$

Example 5:

Evaluate without recourse to aids: $e^{\ln x}$

Solution:

By definition, $\ln x$ represents the number to which e must be raised to produce x. Therefore, in the above expression, e is being raised to that number which will produce x.

ie: $e^{\ln x} = x$

IV SOLVING EXPONENTIAL EQUATIONS

Example 6:

Solve for x correct to 2 decimal places: $e^{-0.6x} = 5$

Solution:

$\ln e^{-0.6x} = \ln 5$ (take natural log both sides)

ie, $-0.6\,x = \ln 5$

$\therefore\ x = \dfrac{\ln 5}{-0.6}$

$= \dfrac{1.6094}{-0.6}$

$= -2.6823$

$\therefore$ correct to two decimal places, $x = -2.68$

Example 7:

Solve for x correct to 2 decimal places: $3^x = 5$

Solution:

Method (i)

$$\log 3^x = \log 5$$

$$x \log 3 = \log 5$$

$$x = \frac{\log 5}{\log 3}$$

$$= \frac{0.6990}{0.4771}$$

$$= 1.4649$$

$\therefore$ $x = 1.46$, correct to 2 decimal places

Method (ii)

$\ln 3^x = \ln 5$ (take natural log of both sides)

$x \ln 3 = \ln 5$ (law 3)

$$x = \frac{\ln 5}{\ln 3}$$

$$= \frac{1.6094}{1.0986}$$

$$= 1.4649$$

$\therefore$ $x = 1.46$, correct to 2 decimal places.

This example has been evaluated using both common and natural logarithms to demonstrate that, regardless of which base is used, the answer will be the same.

Example 8:

The activity of a radioactive source after t seconds is given by:

$$A(t) = A_0 e^{-\lambda t}$$

where A_0 = original activity at t = 0, and λ is the decay constant in s^{-1} (per second)

(a) If A_0 = 9.5 Ci, A(t) = 7.2 Ci, and t = 2 hr, calculate λ.

(b) Using that value of λ, calculate the half life of the radionuclide (ie, the time for the activity to decrease by a factor of 2).

Solution:

Method (i)

(a) $7.2 = 9.5\ e^{-7.2 \times 10^3 \lambda}$, (t = 2 hr = 7.2×10^3 sec)

$\log 7.2 = \log 9.5 - 7.2 \times 10^3\ \lambda \log e$ (common log of both sides)

$$\therefore \quad \lambda = \frac{\log 7.2 - \log 9.5}{-7.2 \times 10^3 \log e}$$

$$= \frac{0.8573 - 0.9777}{-7.2 \times 10^3 \times 0.4343}$$

$$= 3.85 \times 10^{-5}$$

$$\therefore \quad \lambda = 3.85 \times 10^{-5}\ s^{-1}$$

Method (ii)

$$7.2 = 9.5e^{-7.2 \times 10^3 \lambda}$$

$\ln 7.2 = \ln 9.5 - 7.2 \times 10^3 \lambda \ln e$ (nat. log of both sides)

$$\therefore \quad \lambda = \frac{\ln 7.2 - \ln 9.5}{-7.2 \times 10^3 \ln e}$$

$$= \frac{1.9741 - 2.2513}{-7.2 \times 10^3 \times 1}$$

$$= 3.85 \times 10^{-5}$$

$$\therefore \quad \lambda = 3.85 \times 10^{-5}\ s^{-1}$$

NOTE: Whether you use common logs or natural logs, the answer is the same.

(b) Let T be the half-life of the radionuclide

After 1 half-life, $A(T) = .5\ A_0$

ie: $.5\ A_0 = A_0 e^{-3.85 \times 10^{-5}\ T}$

$$\therefore \quad .5 = e^{-3.85 \times 10^{-5}\ T}$$

$$\therefore \ln\ .5 = -3.85 \times 10^{-5}\ T \ln e$$

$$\therefore \quad T = \frac{\ln .5}{-3.85 \times 10^{-5} \times \ln e}$$

$$= \frac{-0.6931}{-3.85 \times 10^{-5} \times 1}$$

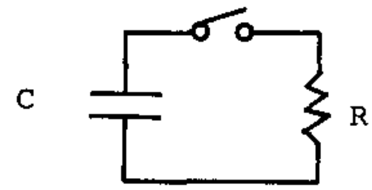
$$= 1.8 \times 10^4\ s$$

$$\therefore \quad T = 1.8 \times 10^4\ s = 5\ h$$

ASSIGNMENT

1. If, at $t = o$, the switch is closed in the circuit illustrated below, the voltage V across the capacitor after t seconds is given by the formula,

$$V(t) = V_o e^{-t/RC}$$



where V_o volts is the original voltage across the capacitor at time $t = o$,

R ohms is the resistance in the circuit, and

C farads is the capacitance of the capacitor.

Find (i) R

(ii) the discharge current, $I(t)$ (recall Ohms Law: $I = \frac{V}{R}$)

if (a) $V_o = 12$ V, $V(t) = 2$ V, $t = 6$ s, $C = 2$ F

(b) $V_o = 1$ V, $V(t) = 0.1$ V, $t = 10^{-3}$s, $C = 200$ μF.

2. A radioactive source decays from 10 Ci to 4.5 Ci in 3.0 hours. Calculate

(a) the decay constant λ in s^{-1}

(b) the half-life in hours

of the source. ($A(t) = A_o e^{-\lambda t}$)

3. Evaluate without recourse to aids:

(a) $\ln \sqrt{e}$

(b) $10^{\log 0.6}$

(c) $e^{2\ln 9}$

(d) $3^{\log_3 4}$

(e) $\log_5 5^{-0.2}$

(f) $2 \log 10^{-5}$

(g) $\log_2 1024$

(h) $2 \log_5 625$

4. A 0.5 μF capacitor, resistor and switch are placed in series in a circuit. The capacitor is charged to a voltage of 12 V when the switch is closed. If the voltage decays to 0.1 V after 2 ms, what is the resistance value in the circuit?

$(V(t) = V_O e^{-t/RC})$

5. In a certain quantity of a radioactive substance, there are 10^{20} radioactive nuclei, each of which will eventually decay by a single disintegration to a stable daughter. If $\lambda = 3.0 \times 10^{-5}\ s^{-1}$, find the time required for the number of radioactive nuclei to decrease to 10^{15}.

$(N(t) = N_O e^{-\lambda t})$

6. Find x, correct to 2 significant figures:

(a) $e^{-1.17x} = 37$	*(g) $\log_3 x = 2.7$
(b) $(1.73)^x = 0.0046$	*(h) $\log_7 x = 4.8$
(c) $3^x = 17$	*(i) $\log_9 x = 2.1$
(d) $e^{0.003x} = 146.2$	*(j) $\log_4 x = 5.3$
(e) $\frac{2^{3x}}{7} = 1.3$	*(k) $\log_{17} x = 16.8$
(f) $e^{-0.3x} = 25$	*(l) $\log_6 x = 7.5$

*If your calculator does not have a Y^X function key, derive an expression for your answer.

7. Express in terms of log X, log Y, and log Z:

(a) $\log \sqrt[4]{X^3 \quad \sqrt[3]{Y^2} \quad Z^5}$

(b) $\log \left[\frac{X^{1/3} \sqrt{Z^5}}{Y^3} \right]^3$

(c) $\log \left[\frac{X^5 \sqrt[4]{Y^7} Z^9}{\sqrt{X^3} \quad Y^3 \quad Z^{3/2}} \right]^{1/3}$

8. Find x (correct to 3 significant figures):

(a) $e^{7.2} = x$

(b) $e^{-3.5} = x$

(c) $e^{0.4} = x - 23.82$

(d) $10^{1.7} = x$

(e) $10^{-4.3} = x$

(f) $10^{0.83} = x + 6$

9. Find x, correct to 2 significant figures:

(a) $7^{3.5} = x$

(b) $3^{-0.7} = x$

(c) $1.4^{0.65} = x$

(d) $6^{4.5} = x$

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