APPENDIX 3 A Short Summary of Orthogonal Functions Used in Chapter 4

Given a countably infinite set of functions: $g_1(x), g_2(x), g_3(x) \cdots g_n(x) \cdots g_m(x) \cdots$, the functions are termed orthogonal in the interval $a \leq x \leq b$ if

$$\int_a^b g_m(x) g_n(x)\, dx = 0 \text{ for } m \neq n. \tag{D.1}$$

A set of orthogonal functions has particular value in the possible representation of an arbitrary function as an infinite series of the orthogonal set in the specified interval. If $f(x)$ denotes the arbitrary function, consider the possibility of expressing it as a linear combination of the orthogonal functions:

$$f(x) = C_1 g_1(x) + C_2 g_2(x) + \cdots + C_n g_n(x) + \cdots + C_m g_m(x) + \cdots$$

$$f(x) = \sum_{n=1}^{\infty} C_n g_n(x). \tag{D.2}$$

The C's are constants to be determined. If the series of Eq. (D.2) is convergent and integrable after multiplication by one of the functions, say $g_n(x)$, then

$$\int_a^b f(x) g_n(x)\, dx = C_1 \int_a^b g_1(x) g_n(x)\, dx + C_2 \int_a^b g_2(x) g_n(x)\, dx + \cdots$$

$$+ C_n \int_a^b g_n^2(x)\, dx + \ldots + C_m \int_a^b g_m(x) g_n(x)\, dx + \cdots.$$

The orthogonality definition given in Eq. (D.1) makes all the integrals on the right side of the above equation vanish except for the one term when $m = n$. Thus

$$\int_a^b f(x)g_n(x)\,dx = 0 + 0 + \cdots + C_n \int_a^b g_n^2(x)\,dx + 0 + \cdots,$$

and the constant C_n may be calculated:

$$C_n = \frac{\int_a^b f(x)g_n(x)\,dx}{\int_a^b g_n^2(x)\,dx}. \tag{D.3}$$

Thus when the function $f(x)$ is given, Eq. (D.3) enables one to calculate the constants, C_n, to be used in the series representation of $f(x)$. These constants are expressed in terms of the given set of orthogonal functions, $g_n(x)$.

A set of functions $[g_1(x), g_2(x), \cdots]$ may form an orthogonal set in the interval $a \le x \le b$ with respect to a weighting factor, $p(x)$, if:

$$\int_a^b p(x)g_n(x)g_m(x)\,dx = 0 \text{ for } m \neq n. \tag{D.4}$$

As before, if an arbitrary function, $f(x)$ can be represented as an infinite series of the functions:

$$\begin{aligned} f(x) &= C_1 g_1(x) + C_2 g_2(x) + \cdots + C_n g_n(x) + \cdots + C_m g_m(x) + \cdots \\ &= \sum_{n=1}^{\infty} C_n g_n(x), \end{aligned}$$

then the constants are given by

$$C_n = \frac{\int_a^b p(x)f(x)g_n(x)\,dx}{\int_a^b p(x)g_n^2(x)\,dx}. \tag{D.5}$$

Some of the othogonal sets of functions used in Chap. 4 will now be discussed as examples.

The Sine and Cosine Functions

Consider the following set of functions in the interval $0 \leq x \leq L$:

$$\sin \frac{\pi x}{L}, \sin \frac{2\pi x}{L}, \sin \frac{3\pi x}{L}, \cdots, \sin \frac{n\pi x}{L}, \cdots.$$

This may also be expressed:

$$\sin \lambda_1 x, \sin \lambda_2 x, \sin \lambda_3 x, \cdots, \sin \lambda_n x, \cdots$$

where

$$\lambda_n = \frac{n\pi}{L}, n = 1, 2, 3, \cdots. \tag{D.6}$$

Now in the interval

$$\int_a^b \sin \lambda_n x \sin \lambda_m x \, dx = \left[-\frac{\sin(\lambda_n + \lambda_m)x}{2(\lambda_m + \lambda_n)} + \frac{\sin(\lambda_n - \lambda_m)x}{2(\lambda_n - \lambda_m)} \right] \tag{D.7}$$
$$= 0 \text{ for } \lambda_m \neq \lambda_n,$$

since

$$\lambda_n = \frac{n\pi}{L}, \lambda_m = \frac{m\pi}{L}.$$

Thus the set of functions in Eq. (D.6) is an orthogonal set. Also, for $m = n$:

$$\int_0^L \sin^2(\lambda_n x) \, dx = \frac{1}{2\lambda_n} (\lambda_n x - \sin \lambda_n x \cos \lambda_n x) \Big|_0^L \tag{D.8}$$
$$= \frac{L}{2}.$$

Thus an arbitrary function, $f(x)$, may, if the series converges, be represented as a series of the functions of Eq. (D.6):

$$f(x) = C_1 \sin \lambda_1 x + C_2 \sin \lambda_2 x + \cdots,$$

or

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \lambda_n x. \tag{D.9}$$

The C_n's wili be, from Eqs. (D.3) and (D.8),

$$C_n = \frac{2}{L}\int_0^L f(x) \sin \lambda_n x \, dx.$$
$$\lambda_n = \frac{n\pi}{L}, n = 1, 2, 3. \tag{D.10}$$

Thus, the function $f(x)$ is representable by the following series:

$$f(x) = \frac{2}{L}\sum_{n=1}^{\infty} \sin \lambda_n x \int_0^L f(x) \sin \lambda_n x \, dx. \tag{D.11}$$

In a similar fashion, one can show that the set of functions

$$\{\cos \lambda_n x\}, \lambda_n = \frac{n\pi}{L}, n = 0, 1, 2, 3, \cdots \tag{D.12}$$

is an orthogonal set in $0 \leq x \leq L$. Also, then, an arbitrary function $f(x)$, may be represented as a convergent series of these functions:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \lambda_n x, \tag{D.13}$$

if

$$A_n = \frac{2}{L}\int_0^L f(x) \cos \lambda_n x \, dx \tag{D.14}$$

with

$$\lambda_n = \frac{n\pi}{L}, n = 0, 1, 2, 3, \cdots.$$

Thus,

$$f(x) = \frac{1}{L}\int_0^L f(x)\, dx + \frac{2}{L}\sum_{n=1}^{\infty} \cos \lambda_n x \int_0^L f(x) \cos \lambda_n x \, dx. \tag{D.15}$$

In many instances in heat conduction problems it may be necessary to express a function as an infinite series of the sines or cosines, such as in Eqs. (D.9) or (D.13), but in which the λ_n's are defined by relations other than that specified by Eqs. (D.6) and (D.12). These characteristic equations defining the λ_n's arise out of the application of the boundary conditions of the particular problem under consideration. One such case is discussed in Chap. 4, wherein one wishes to represent a function as a sine series in the interval $0 \leq x \leq L$ as in Eq. (D.9), when the λ_n's are defined as the roots of the equation:

$$(\lambda_n L)\tan(\lambda_n L) - B = 0 \tag{D.16}$$

$$\text{where } n = 1, 2, 3, \cdots.$$

$$B = \text{constant}.$$

Now, since Eq. (D.16) may be written

$$(\lambda_n L)\sin(\lambda_n L) = B\cos(\lambda_n L)$$

some algebra will show that the integral expressed in Eq. (D.7) will again vanish. Also, Eq. (D.8) gives, instead of $L/2$, that

$$\int_0^L \sin^2(\lambda_n x)\,dx = \frac{L}{2} - \frac{\sin(\lambda_n L)\cos(\lambda_n L)}{2\lambda_n}.$$

Thus, if λ_n is a root of

$$(\lambda_n L)\tan(\lambda_n L) - B = 0,$$

then an arbitrary function, $f(x)$, may be expressed as a sine series

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \lambda_n x, \tag{D.17}$$

where

$$C_n = \frac{\displaystyle\int_0^L f(x)\sin\lambda_n x\,dx}{\dfrac{L}{2} - \dfrac{\sin\lambda_n L\cos\lambda_n L}{2\lambda_n}}. \tag{D.18}$$

Thus, in terms of these constants, $f(x)$ may be represented by

$$f(x) = 2 \sum_{n=1}^{\infty} \sin \lambda_n x \frac{\lambda_n \int_0^L f(x) \sin \lambda_n x \, dx}{\lambda_n L - \sin \lambda_n L \cos \lambda_n L}.$$

Similarly when Eq. (D.16) holds, an expansion in terms of cosines may be made:

$$f(x) = \sum_{n=1}^{\infty} A_n \cos \lambda_n x, \tag{D.19}$$

where

$$A_n = \frac{\int_0^L f(x) \cos \lambda_n x \, dx}{\frac{L}{2} + \frac{\sin \lambda_n L \cos \lambda_n L}{2\lambda_n}}. \tag{D.20}$$

The corresponding representation of $f(x)$ is, then

$$f(x) = 2 \sum_{n=1}^{\infty} \cos \lambda_n L \frac{\lambda_n \int_0^L f(x) \cos \lambda_n x \, dx}{\lambda_n L + \sin \lambda_n L \cos \lambda_n L}.$$

The Bessel Functions

In heat conduction problems in cylindrical coordinate systems, the solutions are often expressed in terms of the Bessel functions (Appendix C). To express an arbitrary function as an infinite series of such functions it will be necessary to show their orthogonality. The Bessel functions are orthogonal with respect to the weighting factor: $p(x) = x$. For example, considering J_0, it will be shown that an arbitrary function may be expressed, in an interval, as a linear combination of the set $J_0(\lambda_1 x)$, $J_0(\lambda_2 x)$, $J_0(\lambda_3 x)$, ..., $J_0(\lambda_n x)$, ..., where the parameters denoted by λ_n are defined, in some way, by the boundary conditions of the problem.

In other words, it will be shown that a function $f(x)$ may be represented in the following way, provided that the λ_n's are properly defined:

$$\begin{aligned} f(x) &= C_1 J_0(\lambda_1 x) + C_2 J_0(\lambda_2 x) + \cdots \\ &= \sum_{n=1}^{\infty} C_n J_0(\lambda_n x) . \end{aligned} \tag{D.21}$$

In order to be able to do this, Eq. (D.4) shows that [for $p(x) = x$] the following condition must be satisfied:

$$\int_a^b x J_0(\lambda_n x) J_0(\lambda_m x)\, dx = 0,\ m \neq n . \tag{D.22}$$

Then Eq. (D.5) shows that the constants C_n are:

$$C_n = \frac{\displaystyle\int_a^b x f(x) J_0(\lambda_n x)\, dx}{\displaystyle\int_a^b x J_0{}^2(\lambda_n x)\, dx} . \tag{D.23}$$

In order to prove the orthogonality condition of Eq. (D.22) and to evaluate the constants given by Eq. (D.23), one needs expressions for

$$\int_a^b x J_0(\lambda_n x) J_0(\lambda_m x)\, dx$$

and

$$\int_a^b x J_0{}^2(\lambda_n x)\, dx .$$

These two integrals may readily be evaluated by repeated "integration-by-parts," utilizing the following formulas resulting from Eq. (C.11) of Appendix C:

$$\frac{d}{dx}[J_0(\lambda_n x)] = -\lambda_n J_1(\lambda_n x), \qquad \frac{d}{dx}[x J_1(\lambda_n x)] = \lambda_n x J_0(\lambda_n x)$$

$$\int J_1(\lambda_n x)\, dx = -\frac{1}{\lambda_n} J_0(\lambda_n x), \qquad \int x J_0(\lambda_n x)\, dx = \frac{x}{\lambda_n} J_1(\lambda_n x) .$$

The results are

$$\int xJ_0(\lambda_n x)J_0(\lambda_m x)\,dx = \frac{x}{\lambda_n^2 - \lambda_m^2}[\lambda_n J_0(\lambda_m x)J_1(\lambda_n x) - \lambda_m J_0(\lambda_n x)J_1(\lambda_m x)] \tag{D.24}$$

$$\int xJ_0^2(\lambda_n x)\,dx = \frac{x^2}{2}[J_0^2(\lambda_n x) + J_1^2(\lambda_n x)]. \tag{D.25}$$

As a particular example consider the set of functions $J_0(\lambda_1 x)$, $J_0(\lambda_2 x)$, ... $J_0(\lambda_n x)$, ... in which the λ_n's are defined in the following way for the interval $0 \le x \le R$. Let the λ_n's be the roots of the equation

$$J_0(\lambda_n R) = 0. \tag{D.26}$$

Examination of the tables of $J_0(x)$ given in Appendix C shows that J_0 has a succession of zeros that differ by an interval approaching 2π as $x \to \infty$. Hence, there are a countably infinite set of the λ_n's defined in Eq. (D.26). For the interval $0 \le x \le R$, Eq. (D.24) reduces to

$$\int_0^R xJ_0(\lambda_n x)J_0(\lambda_m x)\,dx = \frac{R}{\lambda_n^2 - \lambda_m^2}[\lambda_n J_0(\lambda_m R)J_1(\lambda_n R) - \lambda_m J_0(\lambda_n R)J_1(\lambda_m R)].$$

By virtue of Eq. (D.26), $J_0(\lambda_n R) = J_0(\lambda_m R) = 0$ so that

$$\int_0^R xJ_0(\lambda_n x)J_0(\lambda_m x)\,dx = 0.$$

Thus the functions $J_0(\lambda_1 x)$, $J_0(\lambda_2 x)$, ..., are orthogonal in the interval $0 \le x \le R$ if λ_n is a root of Eq. (D.26). To use Eq. (D.23) to obtain the constants of the linear series expansion, then Eq. (D.25) must be evaluated for the particular definition of λ_n.
Thus

$$\begin{aligned}\int_0^R xJ_0^2(\lambda_n x)\,dx &= \frac{R^2}{2}[J_0^2(\lambda_n R) + J_1^2(\lambda_n R)]\\ &= \frac{R^2}{2}[0 + J_1^2(\lambda_n R)]\\ &= \frac{R^2}{2}J_1^2(\lambda_n R).\end{aligned}$$

Summarizing, an arbitrary function may be expressed, in the interval $0 \le x \le R$ as a series of J_0's:

$$f(x) = \sum_{n=1}^{\infty} C_n J_0(\lambda_n x).$$

The constants, C_n, will be given by Eq. (D.23):

$$C_n = \frac{\int_0^R x f(x) J_0(\lambda_n x)\, dx}{\frac{R^2}{2} J_1^2(\lambda_n R)} \tag{D.27}$$

if the λ_n's are the roots of

$$J_0(\lambda_n R) = 0.$$

In a similar fashion it may be shown that the same expression,

$$f(x) = \sum_{n=1}^{\infty} C_n J_0(\lambda_n x) \tag{D.28}$$

may be written in the interval $0 \le x \le R$ if the λ_n's are the roots of

$$J_1(\lambda_n R) = 0. \tag{D.29}$$

In this case, the C_n's are given by

$$C_n = \frac{\int_0^R x f(x) J_0(\lambda_n x)\, dx}{\frac{R^2}{2} J_0^2(\lambda_n R)}. \tag{D.30}$$

As a final example, Sec. 4.6 considers the possibility of expressing an arbitrary function, $f(x)$, as a series expansion in $J_0(\lambda_n x)$, when λ_n is defined as the nth root of the transcendental equation

$$(\lambda_n R) \frac{J_1(\lambda_n R)}{J_0(\lambda_n R)} - B = 0. \tag{D.31}$$

In the latter equation B is a constant. That the functions with λ_n thus defined are orthogonal in the interval $0 \leq x \leq R$ can be seen by substitution of Eq. (D.31) into Eq. (D.24):

$$\int_0^R xJ_0(\lambda_n x)J_0(\lambda_m x)\,dx = \frac{R}{\lambda_m^2 - \lambda_n^2}\left[\lambda_n\left(\lambda_m R\,\frac{J_1(\lambda_m R)}{B}\right)J_1(\lambda_n R) - \lambda_m\lambda_n R\left(\frac{J_1(\lambda_n R)}{B}\right)J_1(\lambda_m R)\right] = 0\,.$$

The fact that this latter equation equals zero results from the definition of the λ_n's (and λ_m's) given in Eq. (D.31). Equation (D.25) yields, then,

$$\int_0^R xJ_0^2(\lambda_n x)\,dx = \frac{R^2}{2}\,[J_0^2(\lambda_n R) + J_1^2(\lambda_n R)]$$

so that Eq. (D.23) gives the C_n's to be

$$C_n = \frac{\dfrac{2}{R^2}\displaystyle\int_0^R xf(x)J_0(\lambda_n x)\,dx}{J_0^2(\lambda_n R) + J_1^2(\lambda_n R)}\,. \tag{D.32}$$