Mathematics - Course 121

PROBABILITY

Recall from the previous lesson that reliability is technically defined as a probability. Therefore, to calculate reliabilities, one requires some knowledge of probability theory. The purpose of this lesson is to provide some background in elementary probability theory, starting right from the basic definition of probability. Readers who feel competent in the objectives specified for this lesson should proceed directly to lesson 121.00-4.

I DEFINITION OF PROBABILITY

a) Geometrical Interpretation

The probability of event E, denoted P(E), is defined by the statement,

$$P(E) = \frac{\text{Number of possible outcomes for which E occurs}}{\text{Total number of possible outcomes}}$$

Note that this simplified definition assumes all possible outcomes are equally probable.

b) Empirical Frequency Interpretation

$$P(E) = \lim_{n \to \infty} \frac{f_E}{n}$$

where f_E is the frequency with which E occurs in n trials. (Read RHS as "limit as n tends to infinity of $\frac{f_E}{n}$.")

Note that the empirical frequency definition is universally applicable, whereas the geometrical definition has limited applicability.

Example 1

A fair coin is tossed. Let H and T represent the events "heads" and "tails" respectively.

Applying the geometrical definition,

$$P(H) = \frac{\text{Number of possible outcomes for which H occurs}}{\text{Total number of possible outcomes}}$$

$$= 1/2$$

July 1981

The frequency definition also applies:

$$P(H) = \lim_{n \to \infty} \frac{\text{Number of times H occurs in n trials}}{n}$$

$$= 1/2$$

The frequency definition says that the fraction of tosses for which heads occurs tends to 1/2 as the number of tosses tends to infinity, and that this limiting value of 1/2 is defined as the probability of obtaining heads on any one toss. Thus, the value of P(H) can be found directly from experiment - hence the term "empirical" in "empirical frequency interpretation" of probability.

Example 2

Two dice are tossed. Let E represent "total number of spots = 8". There are five possible outcomes satisfying E, namely, (2,6), (6,2), (3,5), (5,3), and (4,4). Then,

$$P(E) = \frac{\text{Number of possible outcomes for which total = 8 spots}}{\text{Total number of possible outcomes}}$$

$$= \frac{5}{36}$$

Example 3

A hand of 13 cards is chosen at random from a deck of 52 cards. If E represents "hand contains exactly 2 kings", then

$$P(E) = \frac{\text{Number of possible hands containing exactly 2 kings}}{\text{Total number of possible hands}}$$

$$= \frac{\text{(No. ways to get 2 of 4 kings) x (No. ways to get 11 of 48 non king cards)}}{\text{No. possible combinations of 13 from 52 cards}}$$

$$= \frac{{}_4C_2 \times {}_{48}C_{11}}{{}_{52}C_{13}}$$

$$= \left(\frac{4!}{2!2!} \times \frac{48!}{11!37!}\right) \div \frac{52!}{13!39!}$$

$$= 0.2135$$

NB: A brief derivation of the formula for ${}_nC_r$, the number of possible combinations of n objects chosen r at a time, is placed for reference in an appendix to this lesson.

Example 4

Randomly selected, 1 km lengths of transmission cable are inspected for defects. Suppose E denotes "2 defects in a 1 km length of cable".

Then,

$$P(E) = \lim_{n \to \infty} \frac{\text{Number of 1 km lengths examined which have 2 defects}}{\text{Total number of 1 km lengths examined, n}}$$

Note that the geometrical definition is not applicable in this example.

Example 5

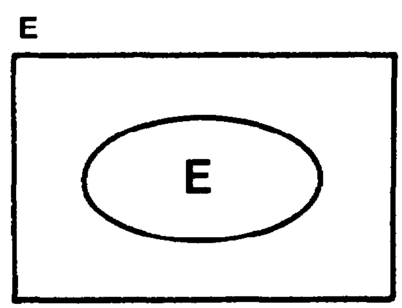
Objects are inspected as they come off an assembly line. Let E denote "object is defective".

$$P(E) = \lim_{n \to \infty} \frac{\text{Number of defective items observed}}{\text{Total number of items examined, n}}$$

Here again, the geometrical definition is not applicable.

II THE VENN DIAGRAM

The Venn diagram is a useful tool in discussing probability, and will be exploited liberally in the semi-intuitive treatment of probability occupying the remainder of this lesson.



The area enclosed by the rectangle represents the universe, S, the set of all possible outcomes of an experiment. The area labelled E, enclosed by the oval, represents the subset of possible outcomes for which the event E occurs. Assuming all outcomes are equally probable,

$$P(E) = \frac{n(E)}{n(S)}$$

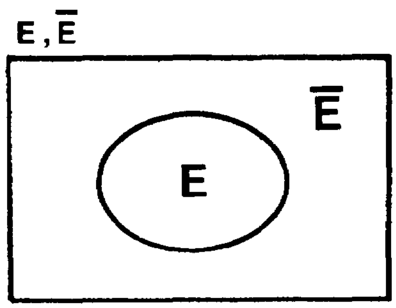
where n(E), n(S) represent the numbers of outcomes in event E and the universe, respectively. This is just a restatement of the geometrical definition of P(E) given earlier. More schematically,

$$P(E) = \frac{\text{(E)}}{\boxed{S}}.$$

The RHS of this expression can represent either $\frac{n(E)}{n(S)}$ or the limit $\lim_{n \to \infty} \frac{f_E}{n}$. Thus the Venn diagram is useful whether or not the geometrical definition of probability applies.

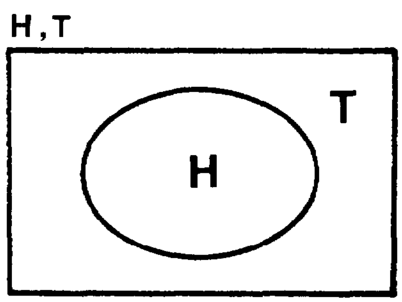
Definition

The complement of event E, denoted $\bar{E}$, (also called "not E"), consists of all possible outcomes in the universe that are not included in E. $\bar{E}$ is shown schematically in the Venn diagram below:



Example 1

The universe for a coin toss is the set of events H,T:

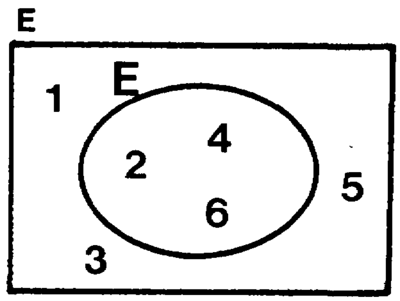


$$P(H) = \frac{n(H)}{n(S)} = 1/2$$

$$P(T) = \frac{n(T)}{n(S)} = 1/2$$

Example 2

For a die toss, the universe consists of the set of outcomes, 1, 2, 3, 4, 5, 6. Let E denote "even number of spots".



$$P(E) = \frac{n(E)}{n(S)}$$

$$= 3/6$$

$$= 1/2$$

III PROBABILITIES OF EVENT COMBINATIONS

The following three examples illustrate the need for formulas to calculate the probabilities of various kinds of event combinations.

Example 1

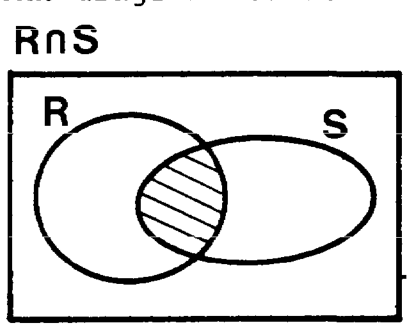
A certain type of nuclear accident involving release of radioactivity to the environment consists of the simultaneous occurrence of events R and S, where R denotes "failure of process system" and S denotes "failure of safety system". Then if E denotes occurrence of this type of accident,

$P(E) = P(R \text{ and } S)$

Definition

The occurrence of both events R and S is called the **intersection** of R and S, denoted $R \cap S$.

The intersection of events R and S is represented by the shaded area on the Venn diagram below:



Example 2

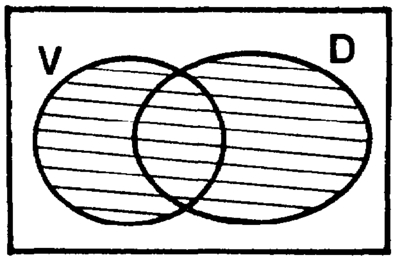
The operation of a certain system can be failed by event D or V or both D and V. Then, if E denotes "system failure",

P(E) = P(V or D or both V and D)

Definition

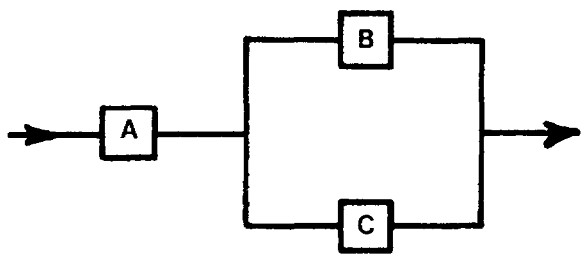
The occurrence of either V or D or both V and D is called the **union** of V and D, denoted $V \cup D$, and is represented by the shaded area in the following Venn diagram:

$V \cup D$



Example 3

With reference to the system illustrated below, let events A, B, C denote failure of components A, B, C, respectively, and E represent system failure.

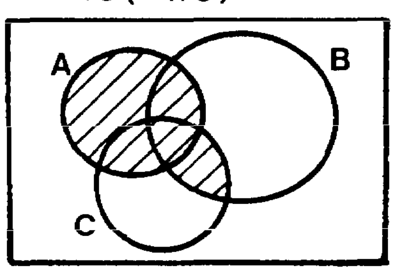


If components B and C are 100% redundant, then the system fails if A fails, or if B and C both fail, or if A, B and C all fail.

$\therefore$ $P(E) = P(A \cup [B \wedge C])$

The Venn diagram for this example is shown below:

$E = A \cup (B \cap C)$



IV PROBABILITY RULES

The remainder of this lesson is devoted to developing 9 probability rules which will be referenced as "PR1", "PR2",..."PR9" throughout the rest of the course. The derivation of each formula will be followed by at least one example of its application. The examples are so simple that they can be verified independently of the newly derived formulas.

1. Intersection of Independent Events

Definition

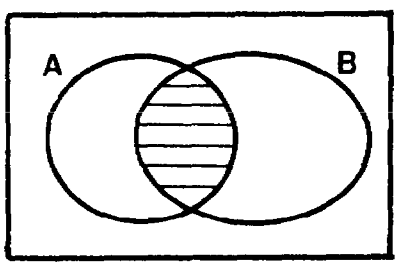
Events A and B are independent if the occurrence of A has no effect on the probability of the occurrence of B, and vice versa.

Example of Independent Events

A coin and a die are tossed. Let H denote heads on the coin and E an even number of spots on the die. Then H and E are independent regardless of the order in which the coin and die are tossed.

Suppose that events A and B in the following diagrams are independent.

$A \cap B$



Then $P(A) = \frac{\textcircled{A}}{\boxed{S}}$ by the basic definition of the Venn diagram.

But since the occurrence of B has no effect on the probability of A's occurring, then the fraction of $\textcircled{B}$ for which A occurs must be the same as the fraction of the universe for which A occurs,

ie, $\frac{\blacklozenge}{\textcircled{B}} = \frac{\textcircled{A}}{\square}$ (*)

Now $P(A \cap B) = \frac{\blacklozenge}{\square}$ (definition of Venn diagram)

$= \frac{\blacklozenge}{\textcircled{B}} \; \frac{\textcircled{B}}{\square}$ (RHS times $\frac{\textcircled{B}}{\textcircled{B}}$)

$= \frac{\textcircled{A}}{\square} \; \frac{\textcircled{B}}{\square}$ (* in RHS) ie, $P(A \cap B) = P(A)\ P(B)$ PR1

Example

Using events E and H as defined above for tossing coin and die,

$P(E \cap H) = P(E)\ P(H)$

$= 3/6 \times 1/2$

$= 1/4$

Check

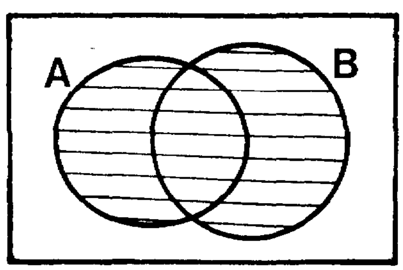
Outcome	H1	H2	H3	H4	H5	H6	T1	T2	T3	T4	T5	T6
E?		✓		✓		✓		✓		✓		✓
H?	✓	✓	✓	✓	✓	✓						
E∩H?		✓		✓		✓						

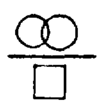
The above table shows that $P(E \cap H) = 3/12$

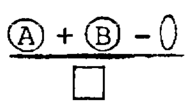
$= 1/4$

2. Union of Events

A ∪ B



$P(A \cup B) =$  (definition of Venn diagram)

$=$ 

ie, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ PR2

Example

A die is tossed. If "A" denotes an "odd number of spots" and "B" denotes "number of spots >3", then,

$P(A) = \frac{1}{2}$; $P(B) = \frac{1}{2}$; $P(A \cap B) = \frac{1}{6}$

and $P(A \cup B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{6}$

$= \frac{5}{6}$

Check

Outcome	1	2	3	4	5	6
A?	✓		✓		✓	
B?				✓	✓	✓
$A \cap B$					✓	
$A \cup B$	✓		✓	✓	✓	✓

NB: A,B are not independent, $\therefore$ $P(A \cap B) \neq P(A)\ P(B)$

3. Union of Independent Events

Substituting PR1 in PR2 gives

$$P(A \cup B) = P(A) + P(B) - P(A)\ P(B) \qquad \text{PR3}$$

Example

Both a coin and a die are tossed.
Let "T" denote "tails" and
"6" denote "6 spots".

Then,

$P(T \cup 6) = P(T) + P(6) - P(T)\ P(6)$

$= \frac{1}{2} + \frac{1}{6} - \frac{1}{2} \times \frac{1}{6}$

$= \frac{7}{12}$

Check

Outcome	H1	H2	H3	H4	H5	H6	T1	T2	T3	T4	T5	T6
T?							✓	✓	✓	✓	✓	✓
6?						✓						✓
T∪6?						✓	✓	✓	✓	✓	✓	✓

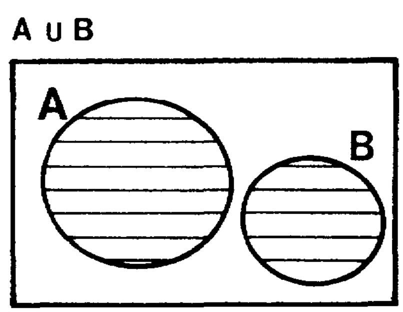
$\therefore P(T \cup 6) = \frac{7}{12}$

4. Union of Mutually Exclusive Events

Definition

A and B are mutually exclusive events if the occurrence of A precludes the occurrence of B, and vice versa, ie, $P(A \cap B) = 0$.

The following Venn diagram shows the union of mutually exclusive events A and B:



Substituting $P(A \cap B) = 0$ in PR3 gives

$$P(A \cup B) = P(A) + P(B) \qquad \text{PR4}$$

Example

A die is tossed.
Let A denote "3 spots" and
B denote "even number of spots".

Since A and B are mutually exclusive,

$\therefore P(A \cup B) = P(A) + P(B)$

$$= \frac{1}{6} + \frac{3}{6}$$

$$= \frac{2}{3}$$

<u>Check</u>

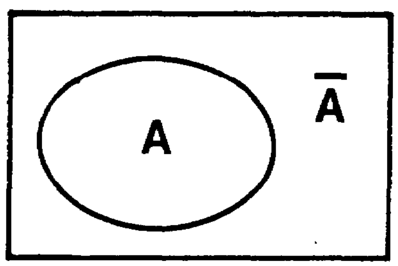
Outcome	1	2	3	4	5	6
A?			✓			
B?		✓		✓		✓
A∪B?		✓	✓	✓		✓

This table shows that P(A B) $= \frac{4}{6}$

$$= \frac{2}{3}$$

5. <u>Union of Complementary Events</u>

$A \cup \bar{A} = S$



Since A,$\bar{A}$ are mutually exclusive,

$\therefore P(A \cup \bar{A}) = P(A) + P(\bar{A})$ by PR4

$$= \frac{\boxed{A}}{\square} + \frac{\boxed{}}{\square}$$

$$= \frac{\square}{\square}$$

$$= 1$$

ie, $\boxed{P(A \cup \bar{A}) = P(A) + P(\bar{A}) = 1}$ PR5

Example

If "A" denotes "odd number of spots", then
"$\bar{A}$" denotes "even number of spots" on toss of single die.
Then,

$$P(A \cup \bar{A}) = \frac{3}{6} + \frac{3}{6}$$

$$= 1$$

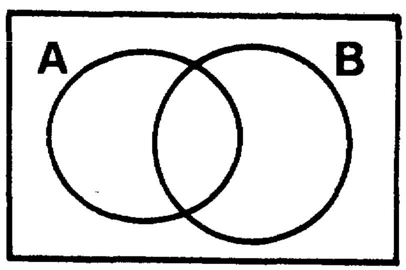
Check

Outcome	1	2	3	4	5	6
A?	✓		✓		✓	
$\bar{A}$?		✓		✓		✓
$A \cup \bar{A}$?	✓	✓	✓	✓	✓	✓

$$\therefore \quad P(A \cup \bar{A}) = \frac{6}{6} = 1$$

6. Conditional Probability

A ,B



The probability of event A conditional that B also occurs is denoted $P(A|B)$.

Then,

$$P(A|B) = \frac{\bigcirc}{\textcircled{B}}$$ (Fraction of B occurrences for which A also occurs.)

$$= \frac{\bigcirc / \square}{\textcircled{B} / \square}$$ (RHS times $\frac{1}{\square} \div \frac{1}{\square}$)

ie, $$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$ PR6

Example

If "A" denotes "3 spots" and
"B" denotes "odd number of spots"
on toss of a single die, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\frac{1}{6}}{\frac{3}{6}}$$

$$= \frac{1}{3}$$

Check

Outcome	1	2	3	4	5	6
B?	✓		✓		✓	
A∩B?			✓			

From this table, one sees that the probability that A occurs, given that B has occurred,

$$P(A|B) = \frac{1}{3}$$

7. Intersection of Dependent Events

The probability of A∩B where A and B are dependent is obtained merely by rearranging PR6:

$$\boxed{P(A \cap B) = P(A|B)\ P(B) = P(B|A)\ P(A)} \qquad \text{PR7}$$

(The second equality holds because $P(A \cap B) = P(B \cap A)$.)

Example

If "A" denotes "odd number of spots" and
"B" denotes "number of spots >3"
on toss of a single die, then,

$$P(A) = \frac{1}{2}\ ;\ P(B) = \frac{1}{2}\ ;\ P(A|B) = \frac{1}{3}\ ;\ P(B|A) = \frac{1}{3}$$

Then,

$P(A \cap B) = P(A|B)\ P(B)$ or $P(B|A)\ P(A)$ by PR7

$= \frac{1}{3} \times \frac{1}{2}$ | $= \frac{1}{3} \times \frac{1}{2}$

$= \frac{1}{6}$ | $= \frac{1}{6}$

Check

Outcome	1	2	3	4	5	6
A?	✓		✓		✓	
B?				✓	✓	✓
A∩B?					✓	

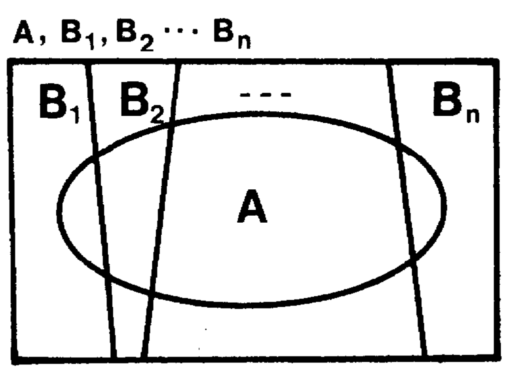
One sees from this table that:

$P(A \cap B) = \frac{1}{6}$

Note that PR1 would give the wrong answer, because A and B in this example are dependent, ie, the occurrence of A affects the probability of the occurrence of B, and vice versa.

8. Baye's Theorem

This theorem is useful in calculating P(A) when event A must occur in conjunction with one of n mutually exclusive events, $B_1, B_2, \ldots, B_n$, as shown schematically in the following Venn diagram:



It is clear from this diagram that

$$A = \sum_{i=1}^{n} (A \cap B_i) \qquad \text{(A equals sum of its parts)}$$

$$\therefore P(A) = \sum_{i=1}^{n} P(A \cap B_i) \qquad \text{(by PR4 since } A \cap B_i \text{ are mutually exclusive)}$$

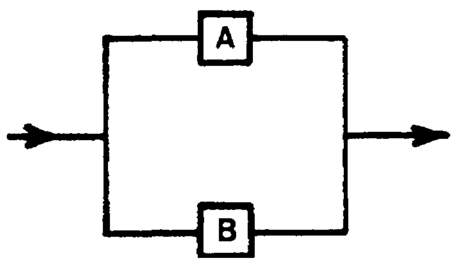
$$= \sum_{i=1}^{n} \frac{P(A \cap B_i)}{P(B_i)} P(B_i)$$

ie, by PR6

$$\boxed{P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)} \qquad \text{PR8}$$

Example 1

A and B are 100% redundant components in the following system:



Calculate the system unreliability, ie, the probability P(S) that the system fails.

Using Baye's Theorem

Let P(A), P(B) represent the probability of failing components A, B, respectively.

Then,

$$P(S) = P(S|A)\ P(A) + P(S|\bar{A})\ P(\bar{A}) \qquad \text{by PR8}$$

Here, the "B_i-events" of PR8 are A, $\bar{A}$ representing component A failure, survival, respectively. Clearly, system failure must occur in conjunction with either A or $\bar{A}$. But $P(S|\bar{A}) = 0$, because the system cannot fail as long as component A survives.

Furthermore, $P(S|A) = P(B)$, because once A is failed, system survival depends entirely on component B.

$\therefore$ $P(S) = P(B)\ P(A)$

Check

The system fails only if both A and B fail simultaneously, ie, $S = A \cap B$,

ie, $P(S) = P(A \cap B)$

ie, $P(S) = P(A)\ P(B)$ by PR1

Example 2

Plant #1 supplies 70% of the quota for a certain item, and 90% of its output meets specs; plant #2 supplies 30% of quota with 80% of its output meeting specs.

a) What is the probability of an item selected at random from the combined stock meeting specs?

b) Given that an item is standard, what is the probability that it comes from plant #2?

Solution

Let events A,B represent manufacture in plants #1,2 respectively, and S denote "item meets specs".

a) $P(S) = P(S|A)\ P(A) + P(S|B)\ P(B)$ by PR8

$= .90 \times .70 + .80 \times .30$

$= .63 + .24$

$= 0.87$

b) $P(B|S) = \frac{P(B \cap S)}{P(S)}$ by PR6

$= \frac{P(B)\ P(S|B)}{P(S)}$ by PR7

$= \frac{.30 \times .80}{.87}$

$= 0.28$

9. Expectation of x, 'E(x)'

Suppose the variable x can take n possible discrete values $x_1, x_2, \ldots x_n$, and that p_1 is the probability that $x = x_1$, p_2 is the probability that $x = x_2$, etc. Then the expectation value of x,

$$E(x) = \sum_{i=1}^{n} x_i p_i \qquad \text{PR9}$$

Example 1

Let x represent the number of spots obtained on tossing a die.

$E(x) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \ldots + 6 \times \frac{1}{6}$

$= 3.5$

Example 2

An insurance company offers to insure a man for $2,000 at a premium of $20 for a period of time for which the probability of his survival is 0.995. Calculate the company's expected profit.

Let x represent the company's gain.

Then E(x) = gain if man survives times probability he survives + gain if man dies times probability he dies

$= \$20 \times 0.995 + (-\$1980) \times 0.005$

$= \$10$

V LESSON SUMMARY

a) Definitions

Independent Events

If A and B are independent events, then the occurrence of A does not affect the probability of the occurrence of B, and vice versa.

Mutually Exclusive Events

A and B are mutually exclusive events if the occurrence of A precludes the occurrence of B and vice versa.

Complementary Events

A and B are complementary events if the non occurrence of A implies the occurrence of B, and vice versa, ie, B = "not A".

Union

The union of events A and B is the occurrence of A or B or both A and B.

Intersection

The intersection of events A and B is the occurrence of both A and B.

Conditional Probability

$P(A|B)$ represents the probability that event A occurs given the condition that event B occurs.

b) Rules for Combining Probabilities

PR1: $P(A \cap B) = P(A)\ P(B)$ (A, B independent)

PR2: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

PR3: $P(A \cup B) = P(A) + P(B) - P(A)\ P(B)$ (A, B independent)

PR4: $P(A \cup B) = P(A) + P(B)$ (A, B mutually exclusive)

PR5: $P(A \cup \bar{A}) = P(A) + P(\bar{A}) = 1$

PR6: $P(A|B) = \frac{P(A \cap B)}{P(B)}$ (Conditional probability for A, B dependent)

PR7: $P(A \cap B) = P(A|B)\ P(B) = P(B|A)\ P(A)$ (A, B dependent)

PR8 Baye's Theorem: If A can occur only in combination with one of n mutually exclusive events B_i, i = 1, 2,...,n,

then,

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

PR9 Expectation Value (Average) If x must take one of n possible values, x_1, x_2,...,x_n, and P_i represents the probability that $x = x_i$, then

$$E(x) = \sum_{i=1}^{n} x_i P_i$$

ASSIGNMENT

1. The probability that a man will be alive in 10 years is 0.8 and the probability that his wife will be alive in 10 years is 0.9. Find the probability that in 10 years:

   a) both will be alive;
   b) only the man will be alive;
   c) only the wife will be alive;
   d) at least one will be alive.

2. Two dice are tossed. What is the probability of:

   a) obtaining a sum of 7?
   b) obtaining no 1?
   c) obtaining one 1?
   d) obtaining at least one 1?

3. Three balls are drawn from an urn containing 5 white balls and 3 black balls. Find the probability that:

   a) all balls are white;
   b) all balls have the same colour;
   c) at least 1 ball will be white.

4. Two numbers are chosen at random from the telephone book. What is the probability that the last digit of each will be different?

5. A family has two children. Each child is as likely to be a boy as it is a girl. What is the conditional probability that both children are girls given that:

   a) the oldest child is a girl and
   b) at least one child is a girl.

6. A die and a coin are tossed and one card is drawn from a deck of 52 cards. What is the probability of obtaining:

   a) a 6, a head and the king of spades;
   b) other than a 6, a head and the king of spades;
   c) an odd number, a tail and a club;
   d) a 6 <u>or</u> a head, and a queen.

7. If a bag contains 7 red balls, 4 black balls and 3 white balls, and 5 are selected at random, what is the probability that the five are 3 red, 2 black and no white?

8. An experiment consists of rolling two dice, one red and one green. Calculate the probability of the following events:

   a) E_1 = rolling a total of 5;
   b) E_2 = rolling a 4 (and any other);
   c) E_3 = rolling a total of 15;
   d) E_4 = rolling a 4 on the red AND a 5 on the green;
   e) E_5 = rolling a 4 on the red OR a 5 on the green.

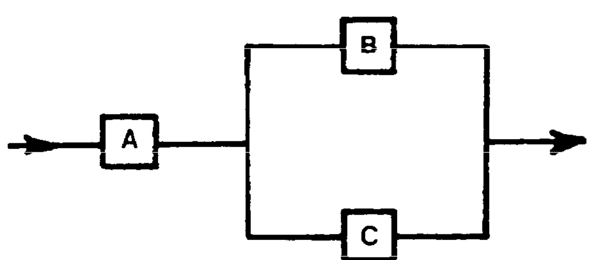
9. In dealing 4 bridge hands, what is the probability that one person will get:

   a) all the cards of one suit?
   b) 4 aces, 4 kings and 4 queens?

10. Consider the following situation in an aircraft. Probability of failure of the engines is .002. Probability of failure of the airframe is .0007. Calculate the probability of a failure within the aircraft. (Note that failures in engines and airframe are not mutually exclusive and should be considered in your calculations, however this factor is extremely small and is usually ignored because the small error it will cause is on the side of safety.)

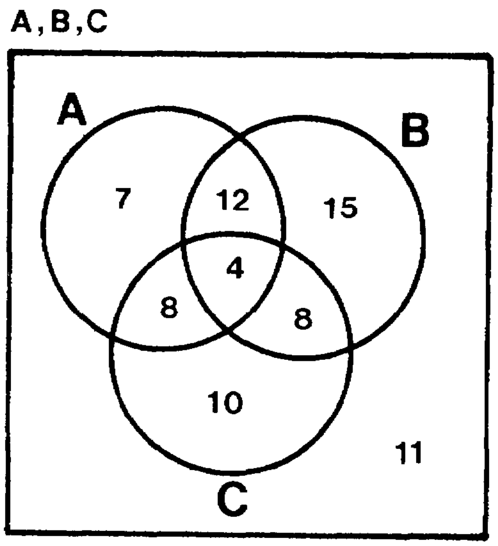
11. A man tosses two fair coins. What is the conditional probability of tossing two heads, given that he has tossed at least one head.

12. In the following system, the component unreliabilities (failure probabilities) for components A, B, C are 0.02, 0.08, 0.10, respectively. If components B and C are 100% redundant, calculate the probability of system failure.



13. A dealer's stock of 75 cars includes cars with power steering (A), compacts (B), and cars with automatic transmissions (C). Using the information given in this figure below find

a) $N(A)$
b) $N(B)$
c) $N(C)$
d) $N(A \cap B)$
e) $N(A \cap C)$
f) $N(A \cap B \cap C)$
g) $N(A \cup B)$
h) $N(B \cup C)$
i) $N(A \cup B \cup C)$
j) $N[B \cap (A \cup C)]$



14. Referring to assignment 13 suppose that one of the cars on the dealer's lot was damaged in a windstorm. Assuming equal probabilities, find the probability that:

a) the damaged car is a compact;
b) the damaged car has power steering;
c) the damaged car is a compact without automatic transmission;
d) the damaged car is not a compact but has an automatic transmission and power steering;
e) the damaged car is a compact, given that it has power steering;
f) the damaged car has automatic transmission, given that it is a compact;
g) the damaged car has power steering or an automatic transmission, given that it is a compact;
h) the damaged car is a compact with automatic transmission, given that it does not have power steering;
i) the damaged car is not a compact, given that it has power steering and an automatic transmission.

15. Three persons work independently at solving a given design problem. The respective probabilities that they solve it are 1/4, 1/3, 1/2. What is the probability that the problem will be solved?

16. A statistics student plays a dice game with his lab instructor. If the student rolls a prime number with one roll of a die, he wins that number of dollars but if a non-prime number occurs, he loses that amount. What is the student's mathematical expectation? Should he be playing the game? Repeat the problem if two dice are used.

17. A pair of fair dice is tossed. Find the probability that at least one of the two numbers is greater than 4.

18. The probability of A hitting a target is 1/2 and the probability that B hits it is 1/4. If they each fire once, what is the probability that the target is hit

    a) twice?
    b) only by A?
    c) only by B?
    d) not at all?
    e) If A can fire only once, how many times must B fire so that there is at least 90% probability that the target will be hit.

19. Two digits are selected at random from the digits 1 through 9. If the sum of the two digits is even, find the probability that both numbers are odd. (Assume digits cannot be repeated.)

20. A player tosses two fair coins. He wins $1.00 if one head appears and $2.00 if two heads appear. On the other hand, he loses $5.00 if no heads appear. What is his expected gain in this game? Should he be playing this game?

21. Two machines produce the total output of a factory. #1 produces 70% and #2, 30% of the output. 5% of the output of machine #1 is defective and 8% from machine #2. If a finished item is selected at random, what is the probability of it being defective?

22. Two dice are tossed together. Let A be the event that the sum of the faces is odd, B the event that at least one face is a one. What is the probability that:

    a) both A and B occur;
    b) either A or B or both occur;
    c) A and not B occurs;
    d) B and not A occurs?

DERIVATION OF ${}_nC_r$ FORMULA

${}_nC_r$, read "n choose r", represents the number of possible different combinations of r objects chosen from n different objects.

Proof that $${}_nC_r = \frac{n!}{r!\,(n-r)!}$$

Definition

n!, read "n factorial", represents the product n(n-1)(n-2)...(3)(2)(1).

Imagine that r objects are drawn one by one from a set of n objects. As each object is drawn, it is placed on one of r squares - the first object is placed on square #1, the second object on square #2, and so on until the rth object is placed on the rth square:

[1] [2] [3] ... [r]

Question

In how many different ways can this be done?

Answer

The first square could have been filled in n different ways, and for each of these, the second square could have been filled in (n-1) ways. For each of the n(n-1) ways of filling the first two squares, the third square can be filled (n-2) ways, and so on. This reasoning leads to the conclusion that the r squares can be filled in a total of

n(n-1)(n-2) ... (n-r+1) ways

However, this is <u>not</u> the correct number of possible different <u>combinations</u>. In fact, this number includes all possible permutations (variations in order) of each possible combination. To see this, consider the following example.

Example

Question

In how many ways can one fill three squares using the digits 1, 2, 3, 4, 5?

Answer

$5(4)(3) = 60$

Question

How many different 3-digit combinations can be formed from the digits 1, 2, 3, 4, 5?

Answer

This is such a simple example that the 10 possible combinations can be listed explicitly:

123, 124, 125, 134, 135, 145, 234, 235, 245, 345.

Question

Why the discrepancy between the number of ways to fill the three squares versus the number of 3-digit combinations?

Answer

The 60 ways to fill the three squares includes 6 permutations for each of the 10 possible combinations. For example, the permutations 123, 132, 213, 231, 312 and 321 all represent different ways to fill the 3 squares, but they all represent the same combination.

Conclusion

Number of 3-digit combinations which can be formed from 5 digits,

$$= \frac{\text{Number of ways to fill 3 squares using 5 digits}}{\text{Number of possible permutations of 3 digits}}$$

$$= \frac{5(4)(3)}{3!}$$

$$= 10$$

Similarly, the number of possible combinations of r objects chosen from n objects,

$$_nC_r = \frac{\text{Number of ways to fill r squares using n objects}}{\text{Number of possible permutations of r objects}}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

Multiplying the RHS by $\frac{(n-r)!}{(n-r)!}$ gives the required formula,

$$\boxed{_nC_r = \frac{n!}{r!(n-r)!}}$$

Example 1

How many different 3-digit combinations can be made from the five digits 1, 2, 3, 4, 5?

Answer

$$_5C_3 = \frac{5!}{3!2!}$$

$$= \underline{\underline{10}}$$

Example 2

How many different hands of 13 cards can be drawn from a deck of 52 cards?

Answer

$$_{52}C_{13} = \frac{52!}{13!39!}$$

$$= \underline{\underline{6.35 \times 10^{11}}} \text{ (about 635 billion!)}$$

L. Haacke