Chapter 3

The Point-Kinetics Equations

Detailed knowledge of the power distribution is very important for the reactor designer, as well as for the reactor operator, who must on a continuing basis monitor the local power distribution against the license limits and ensure the reactor-regulating-system performance. We have seen in the previous chapter how to determine the reference steady-state power distribution for a given reactor configuration by solving the (energy-dependent or multigroup-formalism) time-independent diffusion equation. We have also seen that, in the absence of an external source, the eigenvalue of the homogeneous problem allows us to quantify the difference between the postulated steady state and a "critical" state.

When a perturbation is made to the reactor properties, the steady state no longer holds, and the evolution of the neutron flux must be obtained from the time-dependent diffusion equation. In general, perturbations are not uniform in space. Since the migration length of neutrons is relatively large and their mean lifetime relatively short, the effect of local perturbations on $\phi(r,E,t)$ will quickly propagate. In many cases, if the diffusion length is large and the perturbation not too strong, there is a slight re-adjustment of the flux shape in a few milliseconds, after which the global flux level will increase or decrease, depending on whether the perturbation has increased or decreased k_{eff}. Under such conditions, where the flux shape varies little or slowly, one can predict accurately the evolution of the *total reactor power* as a function of the changes brought by the perturbation to the *average* reactor properties, neglecting completely the shape changes. This is the point-kinetics approximation, which will be discussed in this chapter.

We shall derive the point-kinetics equations in the most general fashion possible, i.e., as a particular case of the factorization method applied to the time-dependent diffusion equation. This factorization transforms the diffusion equation into two equations, one for the *amplitude* and another for the *shape*. The point-kinetics equations are those which allow the calculation of the amplitude.

We have seen in the previous chapter that a steady state is possible in only two distinct situations: either in a critical reactor, or in a sub-critical reactor with an external source. We shall treat these two initial conditions separately, and shall terminate the chapter with the integral formulation of the point-kinetics equations, since this formulation allows the application of several numerical-integration methods. The study of analytical and numerical solutions is left to the following chapters.

3.1 General Formulation

The point-kinetics equations will be derived from the general space-time-kinetics equations, i.e. from the time-dependent diffusion equation (equation 2-35), reproduced here in continuous notation:

$$\boxed{\frac{1}{v}\frac{\partial\phi}{\partial t}(r,E,t) = (F_p - M)\,\phi(r,E,t) + S_d(r,E,t) + S(r,E,t)}$$

with:

$$\begin{aligned} \mathbf{F}_p\phi &= \chi_p(E)\int_0^\infty \nu_p\,\Sigma_f(r,E',t)\,\phi(r,E',t)\,dE' \\ \mathbf{M}\phi &= -\vec{\nabla}\cdot D\vec{\nabla}\phi + \Sigma_t\phi - \int_0^\infty \Sigma_s(r,E'\rightarrow E,t)\phi(r,E',t)dE' \\ S_d &= \sum_k \chi_{dk}(E)\;\lambda_k\;C_k(r,t) \\ S &= \textit{arbitrary independent source} \end{aligned} \tag{3-1}$$

The delayed-neutron source S_d is distributed in the fuel regions (and in the heavy-water reflector in the case of the HWR). It is a function of the local precursor concentration of each delayed-neutron group, C_k, because the emission of delayed neutrons has its origin in the radioactive decay of these precursors. The delayed-neutron source has its own spectrum in energy, χ_{dk}, different from the prompt-neutron spectrum χ_p.

Equation 3-1 is coupled to the precursor equations (2-37), reproduced here:

$$\boxed{\frac{\partial C_k}{\partial t} = -\lambda_k C_k(r,t) + \int_0^\infty dE'\;\nu_{dk}\Sigma_f(r,E',t)\phi(r,E',t)} \tag{3-2}$$

$$(k = 1, 2 \ldots K)$$

The direct solution of equations 3-1 and 3-2, generally in their multigroup form, is an awesome problem. A numerical solution of the space-time problem in its original form requires large resources, given the system dimensions. The problem is even more complex when one takes into account temperature and density feedback, which has an impact on cross sections during a transient. Among the methods which have been introduced to simplify the solution of the equations of space-time kinetics, notable are the factorization methods, which will allows us to introduce point kinetics in a rigorous manner.

3.1.1 Flux Factorization

In order to take advantage of the fact that the flux shape can vary more slowly than the amplitude, we introduce the following factorization:

$$\boxed{\phi(r,E,t) = p(t)\ \psi(r,E,t)} \tag{3-3}$$

In multigroup notation, equation 3-3 takes the form

$$\underline{\Phi} = p(t)\cdot\begin{bmatrix}\psi_1(r,t)\\ \psi_2(r,t)\\ \vdots\\ \psi_G(r,t)\end{bmatrix} \tag{3-4}$$

Note that this factorization is very general. In fact, we have not assumed separability of the space and time variables, since the shape function Ψ is allowed to vary with t, but in principle much more slowly than the amplitude function p. Substitution of this factorized form in the diffusion equation leads to a new equation for the shape function Ψ. After dividing by $p(t)$, we find

$$\boxed{\frac{1}{v}\left(\frac{1}{p(t)}\frac{dp}{dt}\psi(r,E,t) + \frac{\partial\psi}{\partial t}(r,E,t)\right) = \left[F_p - M\right]\psi(r,E,t) + \frac{1}{p(t)}S_d(r,E,t)} \tag{3-5}$$

This equation will be coupled to a second equation for the amplitude $p(t)$, which will be derived below. We note that equation 3-5 is as difficult to solve as equation 3-1, even if $p(t)$ is assumed known. The advantage of this formulation is that in principle the shape Ψ varies much more slowly with t than the amplitude $p(t)$. The discretization in time of equation 3-5 then permits an integration step larger than that used to calculate the amplitude p. In view of the fact that the main numerical effort resides in solving the shape equation, the separate solution of the shape and amplitude equations can lead to substantial machine-time savings. This is the principle inherent in the Improved Quasi-Static approach (Ott, 1969), which we shall discuss again in chapter 8.

The equation for the amplitude is derived by integrating equation 3-1 over the domain (r,E). Since every point in the domain may not have the same importance, we first multiply the diffusion equation by an arbitrary time-independent weighting function $w(r,E)$, defined over the same domain as ϕ.

In analogy with the steady-state production operator (equation 2-72), we define the following operator F in the time-dependent case:

$$F\phi(r,E,t) = (F_p + F_d)\phi(r,E,t) \tag{3-6}$$

We must then make the following observations:

- The operator F_p represents the *instantaneous source of prompt neutrons* (at time t). With factorization 3-3, we get

$$\begin{aligned} F_p\phi(r,E,t) &= p(t)\,\chi_p(E)\int_0^\infty dE'\,\nu_p\,\Sigma_f(r,E',t)\,\psi(r,E,t) \\ &= p\cdot F_p\psi \end{aligned} \tag{3-7}$$

- The operator F_d *does not* represent the instantaneous source of delayed neutrons. The latter arises from the decay rate of the precursors, and is equal to $S_d(r,E,t)$. In fact, F_d is a virtual, rather than real, source of neutrons. It is the source of delayed neutrons which would be found at equilibrium at (r,E) if the reactor were in steady-state with the instantaneous flux $\phi(r,E,t)$. In view of the factorization, we can write

$$\begin{aligned} \mathbf{F}_d\phi(r,E,t) &= p(t)\sum_k \chi_{dk}(E)\overbrace{\int_0^\infty dE'\,\nu_{dk}\Sigma_f(r,E',t)\,\psi(r,E',t)}^{\text{Spatial distribution of the precursor creation rate}} \\ &= p\cdot\sum_k \mathbf{F}_{dk}\,\psi \\ &= p\cdot\mathbf{F}_d\,\psi \end{aligned} \tag{3-8}$$

- In both cases, the sum over the fissionable isotopes is implicit, as in equations 2-21 and 2-38. One must thus understand $\nu\Sigma_f = \Sigma_i(\nu\Sigma_{fi})$.

Let us substitute equation 3-6 into equation 3-1. We find

$$\frac{1}{v}\frac{\partial\phi}{\partial t} = (F - M - F_d)\phi + S_d + S \tag{3-9}$$

Substituting factorization 3-3 into equation 3-9, multiplying by $w(r,E)$ and integrating over the domain, we find the following equation for the amplitude $p(t)$, in the notation introduced in equation 2-94:

$$\left\langle w, v^{-1}\psi\right\rangle\cdot\frac{dp}{dt} + \left\langle w, v^{-1}\frac{\partial\psi}{\partial t}\right\rangle\cdot p = \left\langle w, (F - M - F_d)\psi\right\rangle\cdot p + \left\langle w, S_d\right\rangle + \left\langle w, S\right\rangle \tag{3-10}$$

Let us examine the delayed-neutron source. We have

$$\left\langle w, S_d\right\rangle = \sum_{k=1}^{K}\lambda_k\left\langle w, \chi_{dk}C_k\right\rangle \tag{3-11}$$

Equation 3-11 contains a weighted sum of precursors. We can find an equation or this sum by pre-multiplying equation 3-2 by $\chi_{dk}(E)$ [each term then becoming a function of (r,E,t)], multiplying the result by $w(r,E)$, and integrating over the domain. We find

$$\frac{d\langle w, \chi_{dk} C_k\rangle}{dt} = \langle w, F_{dk}\,\psi\rangle \cdot p(t) - \lambda_k \langle w, \chi_{dk} C_k\rangle \tag{3-12}$$

We shall now define the following scalar quantities to simplify the notation. We first define the relative precursor concentrations $c_k(t)$:

$$c_k(t) = \frac{\langle w, \chi_{dk} C_k\rangle}{\left\langle w, v^{-1}\psi\right\rangle} \tag{3-13}$$

The left-hand side of equation 3-12 contains only the derivative of the numerator of c_k. To obtain the derivative of c_k, we must include the derivative of the denominator, which includes the derivative of the flux shape. Defining the relative shape function $\xi_\psi(t)$ according to

$$\xi_\psi(t) = \frac{\left\langle w, v^{-1}\dfrac{\partial\psi}{\partial t}\right\rangle}{\left\langle w, v^{-1}\psi\right\rangle} \tag{3-14}$$

we find that

$$\frac{1}{\left\langle w, v^{-1}\psi\right\rangle} \cdot \frac{d\langle w, \chi_{dk} C_k\rangle}{dt} = \frac{dc_k(t)}{dt} + \xi_\psi(t) \cdot c_k(t) \tag{3-15}$$

We then define the relative external source $s_e(t)$ as

$$s_e(t) = \frac{\langle w, S\rangle}{\left\langle w, v^{-1}\psi\right\rangle} \tag{3-16}$$

Using definitions 3-13 to 3-16, the amplitude equation 3-10 and the precursor equations 3-12 can then be written in the form

$$\boxed{\frac{dp}{dt} = \left(\frac{\rho}{\Lambda}(t) - \frac{\beta}{\Lambda}(t) - \xi_\psi(t)\right) \cdot p(t) + \sum_{k=1}^{K} \lambda_k c_k(t) + s_e(t)} \tag{3-17}$$

and

$$\boxed{\frac{dc_k}{dt} = \frac{\beta_k}{\Lambda}(t) \cdot p(t) - \left(\lambda_k + \xi_\psi(t)\right) \cdot c_k(t)} \tag{3-18}$$

$(k = 1, 2 \ldots K)$

The following *kinetics parameters* have been introduced, in a notation whose significance will become apparent later:

$$\frac{\rho}{\Lambda}(t) = \frac{\langle w, (F-M)\psi \rangle}{\langle w, v^{-1}\psi \rangle} \tag{3-19}$$

and

$$\begin{aligned}\frac{\beta}{\Lambda}(t) &= \sum_{k=1}^{K} \frac{\beta_k}{\Lambda}(t) \\ &= \sum_{k=1}^{K} \frac{\langle w, F_{dk}\psi \rangle}{\langle w, v^{-1}\psi \rangle}\end{aligned} \tag{3-20}$$

3.1.2 Normalization Constraint

We note that *no approximation has been made to this point.* In place of the time-dependent diffusion equation, coupled to the precursor equations, we have after factorization a system of coupled equations consisting of an equation for the shape function (equation 3-5) and the amplitude equations 3-17 and 3-18. These two systems are entirely equivalent.

The presence of $p(t)$ in equation 3-5 and of $\xi_\psi(t)$ in equations 3-17 and 3-18 ensures the coupling between the equations. We note in addition that the parameters ρ/Λ and β/Λ in the amplitude equations are functionals of the flux shape ψ, solution of equation 3-5. As we shall see in chapter 8, a tremendous numerical advantage can be realized if the shape function varies slowly. Indeed, in the *quasi-static method,* the shape equation is solved over macro-intervals, within which the amplitude equations (i.e., the point-kinetics equations) are separately integrated, using a known shape. The factorization thus introduces an additional degree of freedom, which can be used to simplify the solution of the problem.

We note also that *the relative normalization of p and ψ is arbitrary,* since both these quantities are functions of time. An additional condition is required to fix the relative normalization. That is, in the absence of a normalization constraint, all solutions of the form

$$\begin{aligned}\phi(r,E,t) &= p(t)\cdot\psi(r,E,t) \\ &= \Big(p(t)\ a(t)\Big)\cdot\left(\frac{\psi(r,E,t)}{a(t)}\right) \\ &= \tilde{p}(t)\cdot\tilde{\psi}(r,E,t)\end{aligned} \tag{3-21}$$

where $a(t)$ is an arbitrary function of time must be allowed. Note that the solution of the quasi-static equations without a normalization constraint is in any case possible. Indeed, a unique solution is obtained if a linear variation of the shape function within the macro-intervals is selected and the corresponding value of $\xi_\psi(t)$ is included in the amplitude equation (Monier, 1991).

In order to make the relative normalization unique and to transfer as much as possible of the time variation from ϕ to the amplitude, the following normalization constraint is generally imposed, limiting the variation with time of the shape function at each instant t:

$$\langle w, v^{-1}\psi \rangle = \left\langle \frac{w(r,E)\ \psi(r,E,t)}{v(E)} \right\rangle = K_o \tag{3-22}$$

K_0 is an arbitrary constant. The effect of this *normalization constraint* on ψ is thus to limit the temporal variations of the shape function. Another consequence of equation 3-22 is to remove the function $\xi_\psi(t)$ from the amplitude equations. That is,

$$\xi_\psi(t) = 0 \tag{3-23}$$

Recall that imposing the normalization constraint does not constitute an approximation. The simultaneous solution of equations 3-5, 3-17, and 3-18, together with constraint 3-22 on ψ, affects the value of p but not of the product $\phi = p\psi$, which is conserved. Similarly, the solution of ϕ is independent of the choice for the weighting function w.

3.1.3 Amplitude Equations

The main approximation of point kinetics is the decoupling of the amplitude equations from the shape equation. That is, the amplitude equations 3-17 and 3-18 are solved on the assumption that the shape ψ is known. In the absence of a better prescription for ψ, the approximate shape function $\tilde{\psi}$ is generally selected to be the initial steady-state distribution ψ_0.

The solution to the shape equation 3-5 will thus be replaced as follows:

$$\begin{aligned} \psi(r,E,t) &\approx \tilde{\psi}(r,E,t) \quad \text{(general approximation)} \\ &\approx \psi_0(r,E) \quad \text{(point kinetics)} \end{aligned} \tag{3-24}$$

In view of equation 3-23, the general point-kinetics equation can be written

$$\frac{dp}{dt} = \left(\frac{\rho}{\Lambda}(t) - \frac{\beta}{\Lambda}(t)\right)\cdot p(t) + \sum_{k=1}^{K} \lambda_k c_k(t) + s_e(t) \tag{3-25}$$

and

$$\frac{dc_k}{dt} = \frac{\beta_k}{\Lambda}(t)\cdot p(t) - \lambda_k\, c_k(t) \tag{3-26}$$

where the kinetics parameters defined earlier are now computed with the approximate shape function $\tilde{\psi}$ (equation 3-24):

$$\frac{\rho}{\Lambda}(t) = \frac{\langle w, (F-M)\tilde{\psi}\rangle}{\langle w, v^{-1}\tilde{\psi}\rangle} \tag{3-27}$$

$$\begin{aligned} \frac{\beta}{\Lambda}(t) &= \sum_{k=1}^{K} \frac{\beta_k}{\Lambda}(t) \\ &= \sum_{k=1}^{K} \frac{\langle w, F_{dk}\,\tilde{\psi}\rangle}{\langle w, v^{-1}\tilde{\psi}\rangle} \end{aligned} \tag{3-28}$$

Note that the choice for w was arbitrary in equations 3-19 and 3-20. With the introduction of approximation 3-24, we now allow a difference between the approximate flux shape, $\tilde{\psi}$, used to calculate the amplitude, and the real flux shape, ψ. It will thus be advantageous to choose a weighting function which minimizes the error in $p(t)$ due to the uncertainty in the shape, $\delta\psi = \tilde{\psi} - \psi$. We shall examine the initial conditions of the time-dependent problem to guide us in the choice of w, which will be kept *constant throughout the transient.*

3.2 Common Formulations of the Point-Kinetics Equations

The main approximation of point kinetics consists in decoupling the amplitude equations from the shape equation, where the solution of the coupled system yields the exact solution of the problem. The central approximation of point kinetics thus reduces to the selection of an approximation $\tilde{\psi}(r,E,t)$ for the shape, subject to the normalization constraint 3-22.

We are seeking to determine the evolution of the flux amplitude $p(t)$ in the reactor, springing from the variations of $\phi(r,E,t)$, solution of diffusion equation 3-1 and of the factorization $\phi = p\,\psi$. We shall obtain the amplitude by solving equations 3-25 and 3-26, together making up a system of $(K+1)$ ordinary differential equations. The solution of this system will essentially depend on the time variation of the kinetics parameters 3-27 and 3-28, whose values result from the arbitrary choices for the weighting function w and for the approximation $\tilde{\psi}$ to the real shape ψ.

The choice of weighting function will be dictated by the initial steady-state conditions. We have seen in chapter 2 that only two situations allow a steady state: a critical reactor without an external source, and a subcritical reactor with an external source. We shall examine these two situations separately in the following sections.

3.2.1 Formulation for an Initially Critical Reactor

We shall first assume that the reactor is initially critical and without an external source. Thus, $s(t) = 0$ in equation 3-25. An initial steady state implies that $dp/dt = 0$ and that $dc_k/dt = 0$. Consequently

$$\sum_{k=1}^{K} \lambda_k c_k(0) = \frac{\beta}{\Lambda}(0)\cdot p(0) \tag{3-29}$$

We have therefore verified that the assumption of an initial critical state presupposes that

$$\frac{\rho}{\Lambda}(0) = 0 \tag{3-30}$$

On the other hand, we know that the initial steady state is described by the time-independent diffusion equation 2-78:

$$M_o\phi_o = \lambda_o F_o\phi_o \tag{3-31}$$

where, in principle, $\lambda_0 = 1$ since the reactor is assumed critical.

In practice, even if the critical configuration is known, we allow the calculated value of λ_0 to differ from 1, due to simplifications inherent in the model or to uncertainties in the cross sections which describe the reactor. The initial conditions of the time-dependent problem are then

$$\begin{aligned}\phi(r,E,0) &= p(0)\,\psi_0(r,E,0)\\ &= p_0\,\psi_0(r,E)\\ &= \phi_0(r,E)\end{aligned} \tag{3-32}$$

where ϕ_0 and ψ_0 are both solution of the time-dependent diffusion equation 3-31.

3.2.2 Dynamic Reactivity

The initial flux distribution is thus obtained from the time-independent diffusion equation. The initial eigenvalue can be expressed exactly as the following Rayleigh's quotient (equation 2-117)

$$\lambda_0 = \frac{\left\langle \phi_0^*, M_0\,\phi_0 \right\rangle}{\left\langle \phi^*, F_0\,\phi_0 \right\rangle} \tag{3-33}$$

where ϕ_0^* is the adjoint flux, solution of equation 2-110:

$$M_0^*\phi_0^* = \lambda_0\,F_0^*\phi_0^* \tag{3-34}$$

The concept of static reactivity was introduced in the previous chapter as a measure of the difference between the state of the system and a critical state. The static reactivity, ρ_{s0}, defined as 1 - λ_0, is given by the exact expression

$$\rho_{s0} = \frac{\left\langle \phi_0^*, (F_0 - M_0)\,\phi_0 \right\rangle}{\left\langle \phi_0^*, F_0\,\phi_0 \right\rangle} \tag{3-35}$$

Again, ρ_{s0} is in principle equal to 0 since the reactor is initially critical. The exact value of ρ_{s0} is not significant and should not affect the transient.

The concept of reactivity as a measure of the difference between the steady state of the system and a critical state can be extended to time-dependent problems. We define the *dynamic reactivity* in a manner analogous to equation 3-35:

$$\rho(t) = \frac{\left\langle \phi_0^*, (F - M)\,\phi \right\rangle}{\left\langle \phi_0^*, F\,\phi \right\rangle} \tag{3-36}$$

where $\phi(r,E,t)$ is the time-dependent flux (i.e., the solution sought), and where the elements of operators F and M (i.e., the cross sections) can vary with time. Let ΔF and ΔM be the *changes* to the material properties in the reactor relative to the steady state, as functions of time. We shall write

$$
\begin{aligned}
F &= F_o^{crit} + \Delta F \\
M &= M_o + \Delta M
\end{aligned}
\tag{3-37}
$$

where

$$
F_o^{crit} = \lambda_o F_o \tag{3-38}
$$

We have therefore adjusted the initial fission operator to correspond to a critical reactor. That is in fact the role of the eigenvalue λ_0 in equation 3-31. Correction 3-38 is thus introduced to make the initial dynamic reactivity vanish:

$$
\rho(0) = 0 \tag{3-39}
$$

3.2.3 Choice of Weighting Function

Let us return to equation 3-25. We saw in equation 3-29 that the effect of the parameter β/Λ is initially balanced by the delayed-neutron source, s_{d0}. The evolution of the amplitude will thus be affected mostly by the variation of the parameter ρ/Λ, at least at the beginning.

The choice of weighting function in equation 3-27 is made so as to minimize the error introduced by the uncertainty in the flux shape, introduced by the approximate flux shape ψ. Let us examine the variation of ρ/Λ introduced by the time variation of the shape after $t = 0$, $\Delta\psi = \psi - \psi_0$:

$$
\Delta\left(\frac{\rho}{\Lambda}\right) = \frac{\left\langle w,(F-M)(\psi_o+\Delta\psi)\right\rangle}{\left\langle w,v^{-1}(\psi_o+\Delta\psi)\right\rangle} - \frac{\left\langle w,(F-M)\psi_o\right\rangle}{\left\langle w,v^{-1}\psi_o\right\rangle} \tag{3-40}
$$

Imposing constraint 3-22 leads to

$$
\left\langle w,v^{-1}\Delta\psi\right\rangle = 0 \tag{3-41}
$$

This allows us to write

$$
\begin{aligned}
\Delta\left(\frac{\rho}{\Lambda}\right) &= \frac{\left\langle w,(F-M)\Delta\psi\right\rangle}{\left\langle w,v^{-1}\psi_o\right\rangle} \\
&= \frac{\left\langle w,(\lambda_o F_o - M_o)\Delta\psi\right\rangle}{\left\langle w,v^{-1}\psi_o\right\rangle} + \underbrace{\frac{\left\langle w,(\Delta F - \Delta M)\Delta\psi\right\rangle}{\left\langle w,v^{-1}\psi_o\right\rangle}}_{0(\Delta)^2}
\end{aligned}
\tag{3-42}
$$

We then get

$$
\Delta\left(\frac{\rho}{\Lambda}\right) \approx \frac{\left\langle\left(\lambda_o F_o^* - M_o^*\right)w,\Delta\psi\right\rangle}{\left\langle w,v^{-1}\psi_o\right\rangle} \tag{3-43}
$$

where we have transposed the operator.

We shall therefore choose the initial adjoint flux, ϕ_0^*, for the weighting function:

$$w(r,E) = \phi_0^*(r,E) \tag{3-44}$$

and we then find, in view of equation 3-32:

$$\Delta\left(\frac{\rho}{\Lambda}\right) \approx 0(\Delta)^2 \tag{3-45}$$

This result indicates that the error committed *as a result of using the initial shape* to calculate the kinetics parameter is of second order in the shape error, as long as the initial adjoint flux is used as weighting function. That is,

$$\begin{aligned}\frac{\rho}{\Lambda}(t) &\approx \frac{\langle w,(F-M)\tilde{\psi}\rangle}{\langle w,v^{-1}\tilde{\psi}\rangle} \\ &= \frac{\langle \phi_0^*,(F-M)\psi_0\rangle}{\langle \phi_0^*,v^{-1}\psi_0\rangle} + 0(\Delta)^2\end{aligned} \tag{3-46}$$

We can thus evaluate the kinetics parameter during the transient by using an approximation $\tilde{\psi}$ to the shape, as long as the initial shape ψ_0 is the solution to the time-independent diffusion equation, $\phi(r,E) = p_0 \cdot \psi_0(r,E)$. *The usual approximation in point kinetics is to assume the shape is unchanged from its initial value*:

$$\tilde{\psi}(r,E,t) = \psi_0(r,E) \tag{3-47}$$

Finally, note that the stationary character of the parameter ρ/Λ with respect to arbitrary variations in the shape applies equally well to the dynamic reactivity, whose numerator is identical.

3.2.4 Conventional Forms of the Point Kinetics Equations

While the amplitude equation contains in reality only two kinetics parameters, as we saw earlier, the conventional form of the point-kinetics equations shows three parameters, by explicitly including the dynamic reactivity as one parameter in the equation. This convention goes back to the earliest days of reactor theory, and is very difficult o abandon, as pointed out by Henry (1975).

Let us arbitrarily introduce the following scalar quantity $F(t)$:

$$F(t) = \langle \phi_0^*, F\tilde{\psi}\rangle \tag{3-48}$$

$F(t)$ is a weighted sum of the fission source. Its variation in time is due only to changes in the fission cross sections, provided the shape $\tilde{\psi}$ is restricted to its initial value as in equation 3-47.

Note that the initial value of the production cross sections has been adjusted to make the dynamic reactivity vanish:

$$\begin{aligned} F(0) &= \left\langle \phi_o^*, \lambda_o F_o \psi_o \right\rangle \\ &= F_o \end{aligned} \qquad (3\text{-}49)$$

The value of $F(t)$ depends also on the normalization constraint 3-22 on the initial shape:

$$\left\langle \phi_o^*, v^{-1} \psi_o \right\rangle = K_o \qquad (3\text{-}50)$$

Multiplying and dividing equation 3-46 by $F(t)$ gives

$$\frac{\rho}{\Lambda}(t) = \underbrace{\frac{\left\langle \phi_o^*, (F - M)\tilde{\psi} \right\rangle}{\left\langle \phi_o^*, F\tilde{\psi} \right\rangle}}_{\rho(t)} \cdot \underbrace{\frac{\left\langle \phi_o^*, F\tilde{\psi} \right\rangle}{\left\langle \phi_o^*, v^{-1}\tilde{\psi} \right\rangle}}_{1/\Lambda(t)} \qquad (3\text{-}51)$$

We have in this way separated the parameter ρ/Λ into two components, the dynamic reactivity $\rho(t)$ and a new parameter, $\Lambda(t)$, which we shall call the *mean prompt-neutron lifetime*:

$$\Lambda(t) = \frac{\left\langle \phi_o^*, v^{-1} \phi \right\rangle}{\left\langle \phi_o^*, F\phi \right\rangle} \qquad (3\text{-}52)$$

Treating the parameter β/Λ (equation 3-28) in a similar fashion, we find

$$\frac{\beta_k}{\Lambda}(t) = \underbrace{\frac{\left\langle \phi_o^*, F_{dk}\tilde{\psi} \right\rangle}{\left\langle \phi_o^*, F\tilde{\psi} \right\rangle}}_{\beta_k(t)} \cdot \underbrace{\frac{\left\langle \phi_o^*, F\tilde{\psi} \right\rangle}{\left\langle \phi_o^*, v^{-1}\tilde{\psi} \right\rangle}}_{1/\Lambda(t)} \qquad (3\text{-}53)$$

We have in this way introduced the new parameter $\beta(t)$, which we shall call the *effective delayed-neutron fraction*:

$$\begin{aligned} \beta(t) &= \sum_{k=1}^{K} \beta_k(t) \\ &= \sum_{k=1}^{K} \frac{\left\langle \phi_o^*, F_{dk}\phi \right\rangle}{\left\langle \phi_o^*, F\phi \right\rangle} \end{aligned} \qquad (3\text{-}54)$$

With this separation of parameters, the amplitude equation 3-25 can now be written (in the absence of an external source)

$$\Lambda(t)\frac{dp}{dt}(t) = [\rho(t)-\beta(t)]\,p(t) + s_d(t) \tag{3-55}$$

where we have shown explicitly the delayed-neutron source term, $s_d(t)$, which appears in a form relative to $F(t)$, weighted by the adjoint

$$\begin{aligned} s_d(t) &= \Lambda(t)\cdot\sum_{k=1}^{K} \lambda_k c_k(t) \\ &= \frac{\left\langle \phi_o^*, \sum_k \chi_{dk}(E)\,\lambda_k\, C_k(r,t)\right\rangle}{\left\langle \phi_o^*, F\psi\right\rangle} \end{aligned} \tag{3-56}$$

In the absence of an external source, the conventional form of the kinetics equations is therefore the following:

$$\frac{dp}{dt}(t) = \left[\frac{\rho(t)-\beta(t)}{\Lambda(t)}\right]p(t) + \sum_{k=1}^{K}\lambda_k c_k(t) \tag{3-57}$$

$$\frac{dc_k}{dt}(t) = -\lambda_k c_k(t) + \frac{\beta_k(t)}{\Lambda(t)}\,p(t) \qquad (k=1,2\ldots K) \tag{3-58}$$

where the kinetics parameters are given by equations 3-36, 3-52, and 3-54.

The dependent variables c_k associated with the precursor concentrations are defined in equation 3-13:

$$c_k(t) = \frac{\left\langle \phi_o^*, \chi_{dk} C_k\right\rangle}{\left\langle \phi_o^*, v^{-1}\psi\right\rangle} = \frac{\left\langle \phi_o^*, \chi_{dk} C_k\right\rangle}{K_o} \tag{3-59}$$

Equations 3-58 and 3-59 represent the most frequently used form of the kinetics equations, which we shall label the *c notation*. As pointed out by Ott (1985), it may be preferable to use a different notation for the precursor dependent variables, a notation relative to the fission rate rather than to the constant K_o. From equations 3-48, 3-50, and 3-52 we observe that

$$\begin{aligned} K_o &= F(t)\cdot\Lambda(t) \\ &= F(0)\cdot\Lambda(0) = F_o\cdot\Lambda_o \end{aligned} \tag{3-60}$$

A second notation, which we shall label ζ *notation*, makes use of the following variables:

$$\zeta_k(t) = \frac{\left\langle \phi_o^*, \chi_{dk}\, c_k \right\rangle}{F_o} \qquad (3\text{-}61)$$

The variables c_k and ζ_k are proportional to one another since, from equation 3-60,

$$\zeta_k(t) = \Lambda_o\, c_k(t) \qquad (3\text{-}62)$$

These quantities both reflect the evolution of the delayed-neutron precursors in the reactor. The choice of a different notation leads to a different form of the kinetics equations. The conventional notation (notation c) is the one leading to equations 3-57 and 3-58. In the ζ notation, the delayed-neutron source d in equation 3-56 becomes

$$s_d(t) = \frac{F_o}{F(t)} \sum_k \lambda_k \zeta_k(t) \qquad (3\text{-}63)$$

and the amplitude equations are written

$$\boxed{\begin{aligned} \frac{dp}{dt}(t) &= \left[\frac{\rho(t)-\beta(t)}{\Lambda(t)}\right] p(t) + \frac{1}{\Lambda_o} \sum_{k=1}^{K} \lambda_k \zeta_k(t) \\ \frac{d\zeta_k}{dt}(t) &= -\lambda_k \zeta_k(t) + \frac{F(t)}{F_o} \beta_k(t)\, p(t) \\ &\qquad\qquad\qquad (k = 1, 2 \ldots\ K) \end{aligned}} \qquad \begin{matrix} (3\text{-}64) \\ (3\text{-}65) \end{matrix}$$

3.2.5 Formulation for an Initially Subcritical Reactor

The derivation of the amplitude equations for the case of an initially subcritical reactor is analogous to that for the previous case. The main differences lie in the presence of an external source, $S(\mathbf{r},E,t)$, in the diffusion equation, and in the choice of weighting function, which must be made appropriately.

In the case of a critical reactor, the steady state is a homogeneous problem and the reactivity is expressed as a bilinear quotient (Rayleigh's quotient) with the property of stationarity with respect to arbitrary variations in the flux shape and in the weighting function, provided the weighting function is the solution of the adjoint operator. Since the latter is unique, the definition of the parameters is exact in equations 3-36, 3-52, and 3-54, and the error made in using an approximation $\tilde{\psi}$ to the shape is of second order if the initial shape is the solution ψ_0 to the time-independent diffusion equation.

We must now choose a weighting function w and an approximation $\tilde{\psi}$ to the shape which are appropriate to. We would like to do this while preserving the stationary property of the reactivity, so that the amplitude equation may have the same properties as equation 3-57 or 3-64.

The subcritical steady state implies the presence of an independent source in the reactor. In the case of the critical reactor, the adjoint flux is a natural choice for the weighting function. In the case of the subcritical reactor, the problem is no longer homogeneous and the adjoint flux no longer unique. Indeed, in the "direct" problem, the flux level in the reactor is not arbitrary, but is instead tied to the source level. However, fin the adjoint problem, the adjoint flux depends on the definition of the adjoint source, which is arbitrary from the start. In fact, the adjoint source cannot be tied to the external source, which is by definition independent.

We have therefore

$$(M_o - F_o)\phi_{so} = S_o \qquad \text{(direct)} \tag{3-66}$$

$$(M_o^* - F_o^*)\phi_{so}^* = S_o^* = \Sigma_{det} \qquad \text{(adjoint)} \tag{3-67}$$

where Σ_{det} is a distribution function characteristic of the system. Its choice is arbitrary. It may for instance be a cross section or cross sections permitting the evaluation of some reaction rate in the reactor, or else a detector reading (hence the subscript *det*). In general, a linear functional of the flux can be used (Stacey, 1974):

$$R_{det} = \langle \Sigma_{det}\phi_o \rangle \tag{3-68}$$

The adjoint function φ_{s0} can be used to derive a variational principle for R_{det}, which will be stationary with respect to arbitrary variations in the flux $\delta\phi_0$. Evidently, φ_{s0} is not uniquely defined for a given subcritical reactor. It depends on the detector position or on the definition of the characteristic functional. To solve this problem, it is easier to use as weighting function the adjoint of *the initial problem, rendered homogeneous by the removal of the source.* This homogeneous problem is described by

$$(M_o - \lambda_o F_o)\phi_{\lambda o} = 0 \tag{3-69}$$

The adjoint equation for this homogeneous problem is unique and independent of the external source:

$$(M_o^* - \lambda_o F_o^*)\phi_{\lambda o}^* = 0 \tag{3-70}$$

We note that

$$\phi_{\lambda o}(r,E) \neq \phi_{so}(r,E) \tag{3-71}$$

Indeed, only ϕ_{s0} has physical significance. It is the initial flux distribution in the reactor, in the presence of the source. On the other hand, $\phi_{\lambda 0}$ is a solution of the homogeneous time-independent equation, whose fundamental eigenvalue can be quite different from unity. The distribution $\phi_{\lambda 0}$ is therefore more virtual than real: it is the steady-state distribution one would find in the reactor in the absence of the source, if the mean number of fission neutrons were artificially increased (since $\lambda_0 > 1$). In addition, the normalization of $\phi_{\lambda 0}$ is arbitrary. Note however that the more λ_0 approaches 1, the smaller the source must be for a steady state to exist. It can thus be shown that the non-homogeneous solution tends to the homogeneous solution and that it can be obtained from a perturbation about the critical state (Bruna, 1990). Generally speaking, the homogenous adjoint $\phi^*_{\lambda 0}$ is unique and linearly independent of the initial flux distribution ϕ_{o}.

The previous formulation of the amplitude equations for the critical case is independent of the initial conditions. The equations for an initially subcritical reactor are obtained in precisely the same way, except that the adjoint of the initial fundamental mode, $\varphi_{\lambda 0}$, is used as weighting function in the evaluation of the kinetics parameters 3-36, 3-52, and 3-54:

$$w(r,E) = \phi^*_{\lambda o}(r,E) \tag{3-72}$$

We shall also have to account for the presence of the external source in the amplitude equation. Note that the independent source (equation 3-16) may be written in terms of the prompt-neutron lifetime:

$$s_e(t) = \underbrace{\frac{\left\langle \phi^*_{\lambda o}, S \right\rangle}{\left\langle \phi^*_{\lambda o}, F\psi \right\rangle}}_{s(t)} \cdot \underbrace{\frac{\left\langle \phi^*_{\lambda o}, F\psi \right\rangle}{\left\langle \phi^*_{\lambda o}, v^{-1}\psi \right\rangle}}_{1/\Lambda(t)} \tag{3-73}$$

We have therefore introduced the relative (to the fission rate) external source $s(t)$:

$$s(t) = \frac{\left\langle \phi^*_{\lambda o}, S \right\rangle}{\left\langle \phi^*_{\lambda o}, F\psi \right\rangle} \tag{3-74}$$

For the *initially subcritical* reactor, we find therefore in ζ notation:

$$\frac{dp}{dt}(t) = \left[\frac{\rho(t)-\beta(t)}{\Lambda(t)}\right]p(t) + \frac{1}{\Lambda_o}\sum_k \lambda_k \zeta_k(t) + \frac{s(t)}{\Lambda(t)} \tag{3-75}$$

$$\frac{d\zeta_k}{dt}(t) = -\lambda_k \zeta_k(t) + \frac{F_\lambda(t)}{F_{\lambda o}}\beta_k(t)\, p(t) \tag{3-76}$$

where the parameters ρ, β, and Λ are given by equations 3-36, 3-52, and 3-54, with ϕ_0 replaced by $\varphi_{\lambda 0}$ and $F(t)$ replaced by $F_\lambda(t)$, defined as

$$F_\lambda(t) = \left\langle \phi^*_{\lambda o}, F\psi \right\rangle \tag{3-77}$$

The dynamic reactivity for the initially subcritical reactor can then be written

$$\boxed{\rho(t) \;=\; \frac{\left\langle \overset{*}{\phi}_{\lambda o}, (F-M)\psi \right\rangle}{\left\langle \overset{*}{\phi}_{\lambda o}, F\,\psi \right\rangle}} \tag{3-78}$$

The operators ***F*** and ***M*** are time dependent. Relative to the initial steady-state values, we can define in general

$$\begin{aligned} F &= F_o + \Delta F \\ M &= M_o + \Delta M \end{aligned} \tag{3-79}$$

Note that, in contrast with equation 3-37, it is no longer necessary to adjust the production cross sections to make the initial reactivity vanish. Indeed we accept that the initial reactivity is negative, since the reactor is subcritical. It can easily be shown that the initial value of $\rho(t)$ is equal to the static reactivity (equation 3-35), regardless of the initial shape $\tilde{\psi}_0$ used:

$$\begin{aligned} \rho(0) &= \frac{\left\langle \overset{*}{\phi}_{\lambda o}, (F_o - M_o)\tilde{\psi}_o \right\rangle}{\left\langle \overset{*}{\phi}_{\lambda o}, F_o\tilde{\psi}_o \right\rangle} \\ &= \frac{\left\langle \tilde{\psi}_o, (\overset{*}{F}_o - \overset{*}{M}_o)\overset{*}{\phi}_{\lambda o} \right\rangle}{\left\langle \tilde{\psi}_o, \overset{*}{F}_o\overset{*}{\phi}_{\lambda o} \right\rangle} \;=\; 1 - \lambda_o \\ &= \rho_o \end{aligned} \tag{3-80}$$

As could have been expected, we note that the expression for the reactivity (equation 3-78) does not contain the independent source S, because, in a reactor, reactivity is always defined independently of the source.

We shall now show that the dynamic reactivity (equation 3-77) is stationary with respect to arbitrary variations in the shape function, when these are defined relative to the initial steady-state distribution ψ_{s0}. If this is the case, then it will be advantageous to use *the initial steady-state distribution with source as an approximation of the shape* in the evaluation of the parameters:

$$\begin{aligned} \tilde{\psi}(\mathbf{r},E,t) &= \psi_{so}(\mathbf{r},E) \\ &= \frac{1}{\rho_o}\cdot\phi_{so}(\mathbf{r},E) \end{aligned} \tag{3-81}$$

Let us consider the time variation of the dynamic reactivity. It is written relative to the initial value ρ_0:

$$\rho(t) \;=\; \rho_o \;+\; \delta\rho(t) \tag{3-82}$$

We can write $\delta\rho(t)$ in terms of the variations in the operators by using the exact perturbation formula 2-123 for the reactivity change:

$$\delta\rho(t) = \frac{\left\langle \phi^{*}_{\lambda o}, (\lambda_o \Delta F - \Delta M)\psi \right\rangle}{F_\lambda(t)} \tag{3-83}$$

It can be shown that equation 3-83 is *stationary with respect to arbitrary variations of* ψ *about the initial state* ψ_{s0}, solution of the initial homogeneous problem (that is, the real flux in the reactor with source). Define

$$\psi(r,E,t) = \psi_{so}(r,E) + \Delta\psi(r,E,t) \tag{3-84}$$

By writing explicitly $\Delta F = F - F_0$ and $\Delta M = M - M_0$, equation 3-83 can also be written

$$\boxed{\delta\rho(t) = \frac{\left\langle \phi^{*}_{\lambda o}, (\lambda_o F - M)\psi \right\rangle}{F_\lambda(t)}} \tag{3-85}$$

We substitute equation 3-84 into 3-85 and find

$$\begin{aligned} \delta\rho(t) = \frac{1}{F_\lambda(t)} \Big\{ & \left\langle \phi^{*}_{\lambda o}, (\lambda_o F_o - M_o)\psi_{so} \right\rangle \\ & + \left\langle \phi^{*}_{\lambda o}, (\lambda_o \Delta F - \Delta M)\psi_{so} \right\rangle \\ & + \left\langle \phi^{*}_{\lambda o}, (\lambda_o F_o - M_o)\Delta\psi \right\rangle \\ & + \left\langle \phi^{*}_{\lambda o}, (\lambda_o \Delta F - \Delta M)\Delta\psi \right\rangle \Big\} \end{aligned} \tag{3-86}$$

The first and third terms on the right-hand side cancel, as a result of the choice of weighting function 3-70. The second term constitutes the largest part of the perturbation formula for the reactivity. The fourth term includes differences between the operators *and* between the flux shapes (the flux variation). Each of these differences is 0 initially, and then evolves during the transient. The fourth term is therefore quadratic in nature, and we can conclude that the reactivity formula 3-85 possesses the same property of stationarity as the formula for the initially critical reactor (equation 3-36), as long as

- $\tilde{\psi}$ is indeed the initial flux distribution ψ_{s0}, i.e., the solution of the non-homogeneous problem with source S_0 present;

- the adjoint to the initial homogenous problem (source removed), $\phi^{*}_{\lambda 0}$, is used as weighting function in the calculation of the kinetics parameters.

3.3 Point Model and Interpretation of the Kinetics Parameters

The difference between the two systems of (K+1) ordinary coupled differential equations which we derived for the initially critical reactor stems from the (arbitrary) choice of dependent variable for the precursors. The first form, which derives from the choice of c_k (equation 3-59) as dependent variable, is in much wider use than the second form, which derives from the choice of ζ_k (equation 3-61). We shall nonetheless retain the latter, because it will facilitate the introduction of simplifying assumptions in chapter 5.

The kinetics parameters ρ, β, and Λ, defined above, are all ratios of bilinear products. They are all homogeneous in the flux or in the flux adjoint. Finally, we note that they all include $F(t)$ in the denominator. The choice of the scalar function $F(t)$ in the denominator is purely one of convention. It was made so as to express the dynamic reactivity ρ in the same form as the static reactivity (equation 3-35). On the other hand, note that, after dividing the amplitude equation by $\Lambda(t)$, $F(t)$ disappears from each of the terms. This means that *the solution p(t) of the kinetics equation does not depend on F(t),* nor in fact of any other denominator which could have been chosen in place of $F(t)$.

Indeed, as shown by our general equations 3-25 and 3-26, only two parameters determine the evolution of $p(t)$. These parameters are the two ratios ρ/Λ and β/Λ formed from the three usual parameters. The individual values of ρ, β, and Λ are in principle arbitrary, in view of their denominator. The direct consequence of the conventional approach is that *the three kinetics parameters cannot be individually measured.* In fact, kinetics measurements can provide only values of *ratios* between these parameters (Ott, 1985).

On the other hand, each of the three parameters can be assigned a physical interpretation which allows an easier comparison of the kinetics properties of different reactor types. Historically, the formulation of point-kinetics models has always made use of the three parameters, and we shall follow this convention in the rest of this book.

3.3.1 "Point" Reactor Model

The kinetics parameters ρ, β_k and Λ are integral quantities applying over the entire domain ($\mathbf{r}$,E); they are functionals of the flux shape $\psi(\mathbf{r},E,t)$. If we evaluated ψ by solving equation 3-5, we could calculate the kinetics parameters explicitly, and the evolution of the neutronic power, $p(t)$, could be known precisely from the exact point-kinetics equations.

The main approximation in obtaining the point-kinetics equations from the exact equations for the amplitude $p(t)$ has been to *neglect the time variation of the shape.* One usually uses the initial flux shape to calculate the kinetics parameters:

$$\psi(\mathbf{r},E,t) \approx \psi_o(\mathbf{r},E) = \frac{\phi_o(\mathbf{r},E)}{p_o} \tag{3-87}$$

Two additional approximations are often made in the "point" reactor model:

- The arbitrary denominator $F(t)$ is replaced by its initial value:

$$F(t) \rightarrow F_o = \left\langle \overset{*}{\phi}_o, F_o\psi_o \right\rangle \tag{3-88}$$

- The quasi-steady-state delayed-neutron source, $F_d\phi$, is replaced by its initial value:

$$F_d \approx F_{do} \tag{3-89}$$

These additional simplifications have the following consequences:

1. If we substitute equations 3-87 and 3-88 into equation 3-86, the "exact" formula for the reactivity becomes a perturbation formula of first order (equation 2-124):

$$\rho(t) \rightarrow \rho^{(1)}(t) = \frac{1}{F_o}\left\langle \phi_o^*, (\lambda_o \Delta F - \Delta M)\psi_o \right\rangle \tag{3-90}$$

2. Substituting F_0 for $F(t)$ in equation 3-52, we get

$$\Lambda(t) \rightarrow \Lambda_o = \frac{K_o}{F_o} = \Lambda \tag{3-91}$$

$$\frac{F_o}{F(t)} \rightarrow 1 \tag{3-92}$$

3. Approximation 3-89 leads to the neglect of the (weak) time dependence of the effective delayed-neutron fraction:

$$\beta_k(t) \rightarrow \beta_{ko} = \frac{1}{F_o}\left\langle \phi_o^*, F_{dko}\,\psi_o \right\rangle = \beta_k \tag{3-93}$$

We find in this way the equations for the simplified point model, which is compared in Table 3.1 with the exact point-kinetics equations for the case of the initially critical reactor.

Table 3.1 The Point Kinetiks Equations

	EXACT EQUATIONS	SIMPLIFIED POINT MODEL
c Notation		
$\frac{dp}{dt}(t) =$	$\left[\frac{\rho(t)-\beta(t)}{\Lambda(t)}\right]p(t) + \sum_k \lambda_k c_k(t)$	$\left[\frac{\rho(t)-\beta}{\Lambda}\right]p(t) + \sum_k \lambda_k c_k(t)$
$\frac{dc_k}{dt}(t) =$	$-\lambda_k c_k(t) + \frac{\beta_k(t)}{\Lambda(t)}\,p(t)$	$-\lambda_k c_k(t) + \frac{\beta_k}{\Lambda}p(t)$ (3-94a)
z Notation		
$\frac{dp}{dt}(t) =$	$\left[\frac{\rho(t)-\beta(t)}{\Lambda(t)}\right]p(t) + \frac{1}{\Lambda_o}\sum_k \lambda_k \zeta_k(t)$	$\left[\frac{\rho(t)-\beta}{\Lambda}\right]p(t) + \frac{1}{\Lambda}\sum_k \lambda_k \zeta_k(t)$
$\frac{d\zeta_k}{dt}(t) =$	$-\lambda_k \zeta_k(t) + \frac{F(t)}{F_o}\beta_k(t)\,p(t)$	$-\lambda_k \zeta_k(t) + \beta_k p(t)$ (3-94b)

In the simplified-point-model equations, the only kinetics parameter which varies significantly with time is the dynamic reactivity $\rho(t)$. The other parameters are assumed constant. Note in particular that the precursor equation in the ζ notation becomes independent of Λ. The discussion which follows allows us to evaluate the validity of the approximations which have led to the simplified point model.

3.3.2 Interpretation of the Parameters

a) Dynamic reactivity, $\rho(t)$

The reactivity $\rho(t)$ which appears in equation 3-36 is called the *dynamic reactivity*, in contrast with the static reactivity introduced in the preceding chapter (equation 2-118):

Dynamic reactivity	Static reactivity
$\rho(t) = \dfrac{\left\langle \phi_o^*, (F-M)\psi \right\rangle}{\left\langle \phi_o^*, F\psi \right\rangle}$	$\rho_S(t) = \dfrac{\left\langle \phi_o^*, (F-M)\phi \right\rangle}{\left\langle \phi_o^*, F\phi \right\rangle}$

These two formulas appear identical. Both expressions contain reaction rates weighted by the flux adjoint. However, the essential difference between the two expressions lies in the fact that $\rho(t)$ is calculated from the instantaneous production rate in the reactor, which is a function of the time-dependent flux $\phi(r,E,t)$, as it manifests itself during the transient, whereas ρ_s is calculated from a steady-state flux, where the delayed-neutron source is in equilibrium with the flux. More precisely, the latter is the fundamental mode of type λ of the perturbed system, $\phi(r,E) = \phi_\lambda(r,E)$. In the case of the dynamic reactivity, ϕ_0^* is the adjoint for the initial homogeneous problem, whereas in the case of the static reactivity, it is the adjoint of the flux in the unperturbed system.

The initial adjoint flux is used as weighting function in the derivation of the equations in order to limit the error resulting from the uncertainty in the shape function. As we have seen in the previous chapter, the use of the adjoint flux as weighting function in the calculation of the static reactivity of the perturbed system removes effects of first order in the variation of the flux shape, $\Delta\psi$, caused by the perturbation. The resulting error in ρ_s is thus of second order (i.e., stationary) about the reference (unperturbed) system.

The stationarity properties of the formula for the reactivity are even more important in kinetics problems because

- the spatial distribution, and the spectrum in energy, of the flux can vary with time, $\psi = \psi(r,E,t)$;
- the full time dependence of the shape ψ is in general more difficult to calculate than a few time-independent flux shapes.

This explains the advantage of writing the dynamic reactivity in the same form as the static reactivity (by dividing the reaction rates by $F(t)$ and weighting with the adjoint).

The dynamic reactivity may be considered the most important kinetics parameter, because its variation is usually the source of changes in neutronic power. Indeed, in contrast with the other parameters, the reactivity contains the neutron loss term, which is associated with control mechanisms and which is very sensitive to temperature effects. Reactor perturbations giving rise to transients are thus generally caused by variations in ρ.

We note also that reactivity has no units, since it is a ratio of reaction rates. Its value lies in the following interval:

$$-\infty < \rho(t) < 1{,}0 \tag{3-95}$$

In the positive domain, when the reactor is supercritical, the value of ρ is normally very small for a power reactor, because the reactor is not allowed to stray far from criticality. Usually, positive values of reactivity are of the order of 0.001 or less. For this reason, reactivity is often expressed in *milli-k* (mk), the value being obtained by dividing ρ by 1000. Another unit, often used in Europe, is the *pcm* (part per hundred thousand).

b) Effective Delayed-Neutron Fraction, $\beta(t)$

As its name suggests, the "effective" delayed-neutron fraction $\beta(t)$ is linked to the physical constants of each fissionable isotope which measure the fraction of fission neutrons produced by the decay of certain fission products. We observe indeed that the parameter $\beta(t)$ in equation 3-54 would reduce to the physical delayed-neutron fraction only if the reactor were infinite and uniform and contained only one fissionable isotope.

The following points are worth remembering:

- The operator F_{dk} in the numerator of equation 3-54 implicitly includes a sum over all fissionable isotopes. Based on the (fission) reaction rates, the effective value $\beta(t)$ thus takes into account the distribution of fissionable material in the reactor:

$$F_{dk}\psi(r,E,t) = \sum_i \chi_{dki}(E)\int_0^\infty dE'\, \nu_{dki}\, \Sigma_{fi}(r,E',t)\, \psi(r,E',t) \tag{3-96}$$

Since the delayed-neutron fraction varies significantly from one fissionable isotope to another, *the effective delayed-neutron fraction can vary with fuel burnup*. For example, in a CANDU reactor, most of the fresh fuel in the initial core contains only natural uranium*, with thermal fissions in ^{235}U amounting to 97% of all fissions. After several months at full power, when the on-power refuelling rate has reached equilibrium, a large number of fuel bundles contain plutonium isotopes (mostly ^{239}Pu). In the "equilibrium core", about half of all fissions are in the fissionable plutonium isotopes. Since $\beta_{Pu\text{-}239} < \beta_{U\text{-}235}$, there is a *significant decrease in the effective delayed-neutron fraction in a CANDU reactor as the reactor approaches refuelling equilibrium* (~6 full-power months after startup). During this period, the effective fraction in a CANDU 6 decreases typically from 0.0076 to 0.0059. A smaller decrease will be observed in light-water reactors (^{235}U fissions remain the largest part).

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* A small number of depleted-fuel bundles may be used.

- Weighting by the adjoint is essential because, in addition to weighting the spatial distribution of delayed-neutron precursors, the energy dependence of the adjoint weights the delayed-neutron emission spectrum appropriately. We can judge the importance of this correction by comparing in an approximate way two very different systems, for example fast-neutron reactors and thermal-neutron reactors.

To simplify, let us assume that the adjoint is separable:

$$\phi_o^*(\mathbf{r},E) \approx \phi_o^*(\mathbf{r})\,\varphi_o^*(E)$$

Substituting this approximation in equation 3-54, we find

$$\beta_k(t) \approx \frac{\displaystyle\int_0^\infty dE\,\chi_{dk}(E)\varphi_o^*(E)\int_V dV\,\phi_o^*(r,t)\int_0^\infty dE'\,\nu_{dk}\Sigma_f(r,E',t)\,\psi(r,E',t)}{\displaystyle\int_0^\infty dE\,\chi(E)\varphi_o^*(E)\int_V dV\,\phi_o^*(r,t)\int_0^\infty dE'\,\nu\Sigma_f(r,E',t)\,\psi(r,E',t)} \qquad (3\text{-}97)$$

which may be simplified further by writing

$$\beta_k(t) \approx \underbrace{\gamma_{dk}}_{\substack{\text{Spectral}\\ \text{correction}}} \cdot \underbrace{\bar{\beta}_k(t)}_{\substack{\text{Average}\\ \text{over space}}} \qquad (3\text{-}98)$$

The spectral correction γ_{dk} takes into account the importance of the delayed-neutron spectra relative to the total spectrum:

$$\gamma_{dk} = \frac{\displaystyle\int_0^\infty dE\quad \chi_{dk}(E)\varphi_o^*(E)}{\displaystyle\int_0^\infty dE\quad \chi(E)\varphi_o^*(E)} \qquad (3\text{-}99)$$

We note:

Fast reactors	Thermal reactors
$0.80 < \gamma_{dk} < 0.85$	$1.0 < \gamma_{dk} < 1.25$

The selection and location of fissionable and fertile materials and of the moderator in the reactor therefore has a great influence on the neutron spectrum, and consequently on the effective delayed-neutron fraction.

For example, in a fast reactor, the average importance φ_0^* of the delayed neutrons is 18% smaller than that of all fission neutrons (Ott, 1985). In particular, the energy of delayed neutrons at emission is in general lower than the fast-fission threshold in ^{238}U, which contributes more to the overall fission rate than in a thermal reactor. In addition, the value of the reproduction factor η increases with the incident-neutron energy, which tends to confer greater importance to χ than to χ_d. The reproduction factor is defined as

$$\eta(E) = \frac{\nu(E)\,\sigma_f(E)}{\sigma_a(E)} \qquad (3\text{-}100)$$

The spectral correction therefore reduces the effective value of β in fast reactors. This effect is compensated in part by the spatial weighting when the reactor is heterogeneous (core and mantle). Indeed, with the fertile material concentrated in the mantle (to increase the production of ^{239}Pu), the neutron spectrum is softer at the core-mantle interface than at the center of the core. Consequently, the rate of fast fission in ^{238}U relative to the fission rate in ^{239}Pu, R_{f8}/R_{f9}, is higher at the centre ($\approx$ 14%) than at the core periphery ($\approx$ 8%) (Ott, 1985). Since the importance ϕ_0 is also greater at the centre, and since the delayed-neutron fraction is much greater in ^{238}U than in ^{239}Pu, *the spatial weighting by the adjoint tends to increase the effective value of β in a fast reactor.*

In the case of *thermal reactors*, the spectral correction tends to increase the effective value of β. The adjoint spectrum φ_0 (as well as the flux) increases with decreasing E, as one approaches the thermal-fission domain. Note that delayed neutrons have a greater importance than prompt neutrons in a thermal reactor, because of a greater probability that delayed neutrons will escape resonance absorption or leak out of the system as they are slowed down, since they are emitted with a lower energy than prompt neutrons. *The spectral correction is therefore greater than 1.0 in thermal reactors.*

This effect is more important in light-water reactors (REP, PWR) than in heavy-water or graphite reactors. The latter are larger in size, and fast leakage is relatively smaller. In addition, fast fission is more important in them (in CANDU, $R_{f8}/R_{ftot} \approx 3\%$), which tends to decrease $\beta(t)$, because delayed neutrons appear at energies below the fast-fission threshold in ^{238}U. The conclusion is that *adjoint weighting is essential to obtain an accurate value for the effective delayed-neutron fraction in light-water reactors.* This effect is less important in CANDU reactors.

We finally note that *the time dependence of $\beta(t)$ is generally weak,* since it has its origin mostly in effects of burnup or of refuelling. Since these effects manifest themselves in both the numerator and in the denominator, the time variations of the fission cross sections will tend to cancel. On the other hand, if the fuel is non-uniformly distributed in the reactor, with zones of significantly different fissionable-isotope concentration, a large variation in the flux shape ψ can significantly change the value of β, which must be evaluated as a function of t. Since the main simplifying assumption of point kinetics is to neglect changes in shape (equation 3-87), the additional assumption 3-93 appears justified for the simplified point model.

c) Prompt-Neutron Lifetime, $\Lambda(t)$

The last parameter, $\Lambda(t)$, is usually called the *prompt-neutron lifetime*:

$$\Lambda(t) = \frac{\left\langle \phi_o^*, v^{-1}\psi \right\rangle}{\left\langle \phi_o^*, F\psi \right\rangle} = \frac{K_o}{F(t)} \tag{3-101}$$

As was the case for $\beta(t)$, the definition of $\Lambda(t)$ is conventional and its interpretation stems form the arbitrary choice of $F(t)$ in the denominator. In contrast to ρ and β, $\Lambda(t)$ has units of time, and is generally expressed in seconds (s).

The name of this kinetics parameter goes back to elementary theory, which considers successive neutron generations. The reason for the expression *prompt-neutron lifetime* for Λ is that this parameter measures the average length of time between the disappearance of a neutron (by absorption or leakage out of the system) and the appearance of a neutron of the 2nd-following generation. This time is not 0, since the fission neutron must traverse a certain distance (at speed v) before being absorbed by a fuel nuclide. If Λ were 0, all the energy available from the fissionable material in the reactor would be released instantaneously.

Let us first consider a uniform and infinite multiplying medium. The quantity $1/\Sigma_f$ is the mean free path for fission, i.e., the average distance travelled by a neutron in the medium between its birth and a subsequent fission. Dividing by the neutron speed:

$$\frac{1}{v}\cdot\frac{1}{\Sigma_f} = \Delta t_f$$

we find the time interval Δt_f between the birth of the neutron and the fission which it causes. Since each fission produces ν neutrons, the prompt-neutron lifetime can be obtained by dividing by ν:

$$\Lambda \approx \frac{\Delta t_f}{\nu} = \frac{1}{v\,\nu\Sigma_f}$$

This simplified interpretation can be generalized to the finite reactor. Let us consider the case of a critical reactor in steady state. The fission-neutron source is then simply

$$R_{fo} = p_o F_o = \frac{p_o K_o}{\Lambda_o} \tag{3-102}$$

The numerator includes the shape normalization constant K_o. Note that

$$p_o K_o = \left\langle \phi_o^*, v^{-1}\phi_o \right\rangle$$

is a weighted measure of the neutron population in the system, proportional to the *neutronic power*. If $\phi_0 = 1$, we find the total number of neutrons (at critical equilibrium) in the reactor, N_o.

By considering fission as the event separating two successive generations of neutrons, we can say that the fission source can be obtained as follows:

$$R_{fo} = \frac{N_o}{\Lambda_o} = \frac{\text{Total number of neutrons in the system}}{\text{Average time between the birth of a neutron (prompt or delayed) and its eventual elimination (via absorption or leakage)}}$$

from which the name ***neutron lifetime*** for Λ_0. Since the reactor is not necessarily critical, the flux can vary as a function of time. By analogy, we say that $\Lambda(t)$ represents the mean prompt-neutron lifetime (since the precursor delay is not taken into account). In a sense, $\Lambda(t)$ is a measure of the effective lifetime of the neutron population at time t.

One must not place too great an importance on the physical interpretation of Λ, which rests on the concept of neutron generation, itself poorly defined. Let us remember, more importantly, that $\Lambda(t)$ is the parameter which multiplies the time derivative of the amplitude in the kinetics equation 3-55. Consequently, its value is crucial to the prompt reactor period, i.e., it influences directly the response time of the reactor in strong perturbations, in which prompt neutrons dominate.

$\Lambda(t)$ can vary between 10^{-3} s for thermal reactors and 10^{-7} s for fast reactors. As was the case for the delayed-neutron fraction, *the time dependence of* $\Lambda(t)$ *is often negligible.* On the other hand, if the perturbation in question affects only the fission source, or else if the flux shape ψ varies rapidly during the transient, it is necessary to take into account the time variation of $\Lambda(t)$.

3.4 Integral Formulation of the Equations

The kinetics equations can also be written in an integral form which lends itself to several numerical integration methods or to certain approximations particularly useful in the treatment of temperature feedback (Lemay, 1985). We shall therefore conclude this chapter by presenting the integral form of the kinetics equations, attributed to Keepin and Cox (Keepin, 1960).

We shall retain the assumption of the simplified point model, in c notation. Consider first the precursor equation (3-58). For group-*k* precursor, it can be written

$$\frac{d}{dt}\left[c_k(t)\,e^{\lambda_k t}\right] = \frac{\beta_k}{\Lambda}\,p(t)\,e^{\lambda_k t} \tag{3-103}$$

Integrating this equation from t_0 to t, we find

$$c_k(t) = c_k(t_o)\,e^{-\lambda_k(t-t_o)} + \int_{t_o}^{t} dt'\,\frac{\beta_k}{\Lambda}\,p(t')\,e^{-\lambda_k(t-t')} \tag{3-104}$$

In the limit $t_0 \rightarrow -\infty$, the term containing $c_k(t_0)$ vanishes, because the precursors decay in a finite time. We find therefore an expression for $c_k(t)$ which contains implicitly the initial conditions:

$$c_k(t) = \int_{-\infty}^{t} dt'\,\frac{\beta_k}{\Lambda}\,p(t')\,e^{-\lambda_k(t-t')} \tag{3-105}$$

With steady-state initial conditions for $t < 0$, we find simply

$$c_k(t) = c_{ko}\,e^{-\lambda_k t} + \frac{\beta_k}{\Lambda}\int_{0}^{t} dt'\,p(t')\,e^{-\lambda_k(t-t')} \tag{3-106}$$

with

$$c_{ko} = \frac{\beta_{ko}}{\lambda_k \Lambda_o}\,p_o \tag{3-107}$$

We now introduce $c_k(t)$ in the equation for $p(t)$, by writing $u = t - t'$:

$$\boxed{\frac{dp}{dt}(t) = \left[\frac{\rho(t)-\beta}{\Lambda}\right]p(t) + \frac{\beta}{\Lambda}\int_{0}^{\infty} du\,D(u)\,p(t-u)} \tag{3-108}$$

The kernel $D(u)du$ is the probability that a delayed neutron is emitted between u and $u + du$, for a fission occurring at $u = 0$:

$$D(u) \equiv \sum_{k} \frac{\lambda_k \beta_k}{\beta}\,e^{-\lambda_k u} \tag{3-109}$$

The integro-differential equation 3-108 can be transformed into an integral equation of the Volterra type by substituting equation 3-106 in equation 3-57 and integrating the result by parts:

$$\Lambda(t)\frac{dp}{dt}(t) = \rho(t)\,p(t) - \int_{0}^{t} du\,\frac{dp}{du}(u)\,f(t-u) \tag{3-110}$$

where

$$f(t) = \sum_k \beta_k e^{-\lambda_k t} \tag{3-111}$$

The Laplace transform is defined as

$$\mathcal{L}\{f(t)\} \equiv F(s) = \int_0^{\infty} dt\ f(t)\, e^{-st} \tag{3-112}$$

We have therefore

$$F(s) = \sum_k \frac{\beta_k}{s+\lambda_k} \tag{3-113}$$

The transform of equation 3-110 gives

$$\Lambda\left[s\,P(s)-p_o\right] = \mathcal{L}\{\rho(t)p(t)\} - F(s)\cdot\left[sP(s)-p_o\right] \tag{3-114}$$

since, according to the convolution theorem

$$\mathcal{L}\left\{\int_0^t du\ f_1(t-u)\ f_2(u)\right\} = F_1(s)\cdot F_2(s) \tag{3-115}$$

We can now isolate *P*(*s*):

$$P(s) = \frac{p_o}{s} + Z(s)\cdot\mathcal{L}\{\rho(t)\,p(t)\} \tag{3-116}$$

where the function *Z*(*s*) is defined as

$$Z(s) = \frac{1}{s[\Lambda + F(s)]} = \frac{1}{s\left[\Lambda + \sum_k \frac{\beta_k}{s+\lambda_k}\right]} \tag{3-117}$$

Taking the inverse transform of equation 3-116, we obtain finally the *integral formulation of the kinetics equations*:

$$\boxed{p(t) = p_o + \int_0^t du\ z(t-u)\,\rho(u)\,p(u)} \tag{3-118}$$

We note that the function *z*(*t*) is the inverse transform of *Z*(*s*), the transfer function of the reactor, to which we shall return in the next chapter.

z(*t*) can easily be found from equation 3-117. Let *K* be the number of delayed-neutron groups. The expansion in partial fractions can be written

$$Z(s) = \sum_{n=1}^{K+1} \frac{A_n}{s+\omega_n} \quad \Rightarrow \quad z(t) = \sum_{n=1}^{K+1} A_n\, e^{-\omega_n t} \tag{3-119}$$

where the ω_n are the (*K* + 1) roots of the characteristic polynomial, the denominator of equation 3-117. We shall return in the next chapter to the calculation of the coefficients A_n.

3.5 Conclusion

In this chapter, we have introduced the point-kinetics equations from the general equations for diffusion by making a certain number of simplifying assumptions. The most important was to assume that the perturbation introduced into the reactor affects the flux shape little and that its effect is mostly on the amplitude, associated with the neutronic power.

This approximation is very restrictive, and the point-kinetics model has relatively limited application. In order to find a realistic time-dependent solution in practical cases, it will be necessary to solve the problem in space-time. We shall return to this important aspect in chapter 8.

The reactor regulating system (RRS) typically consists of several control mechanisms (absorbers) arrayed throughout the reactor. The relative movement of these devices allows a reference flux distribution to be maintained in the reactor, while their average movement performs global control. The aim of the RRS is thus generally to

- maintain the reactor critical for operation at constant power;
- provide small changes in reactivity to vary the power;
- ensure spatial control of the power distribution to avoid xenon oscillations from developing (Appendix A).

Several transients associated with global reactor control can thus be studied with point kinetics, since flux distortions will be small or localized. In many cases, the transient can be predicted with accuracy from a given reactivity variation. Even for fast transients, point kinetics allows us to more easily study temperature and void feedback, phenomena which must be included in fast transients but which cannot easily be treated in space-time. Feedback will be discussed in chapters 6 and 7.

In the next chapter, we shall start to study situations where reactivity is constant but can change suddenly. The following chapter will deal with the effect of a gradual change in reactivity, conforming more closely to real displacements of reactivity devices in the reactor.

While the point model is relatively simple and reduces the general problem to a coupled system of $(K + 1)$ ordinary differential equations, the numerical solution of the kinetics equations poses particular problems. Indeed, the time scale associated with prompt neutrons is much shorter than those associated with the delayed-neutron precursors. This particular property of the kinetics equations can lead to the failure of many numerical schemes. This aspect will be discussed in somewhat greater detail in chapter 7.